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RESEARCH MEMORANDUM

AN ANALYSIS OF THE ROTATIONAL MOTION OF A BODY DURING RE-ENTRY

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The dynamical equations of motion are formulated for a body during re-entry into the atmosphere. With the assumption that the translational and rotational degrees of freedom may be treated independently, an analysis is undertaken of the oscillatory motion of a re-entry vehicle which is spinning about its longitudinal axis. The influence of the spin rate and the static margin upon the stability of the transient solution of the equations of motion is considered, and it is concluded that satisfactory dynamical behavior of the re-entry body may be anticipated if:

~~o~~ the static margin of the vehicle is less than zero,

~~o~~ the spin rate is in the range 0.5 to 2 radians per second,

~~o~~ the lift curve has a slope equal to or greater than zero.

An examination of the precessional and nutational motion of the vehicle is also made, and the effect of initial conditions upon these modes of oscillation is indicated. The influence of spin rate and static margin variations upon the forced solutions of the equations of motion is considered, and the following conclusions are drawn:

- o spin rates on the order of 0.5 to 2 radians per second effectively 'average out' re-entry-body asymmetries.
- o for a spin rate of one radian per second the lateral displacement of the re-entry body from the original trajectory plane is considerably less than a thousand feet at impact if the static margin is of the order of minus six inches or less.

Finally, a brief analysis is made of the planar re-entry case, and the results are compared to those obtained by previous investigators.

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SYMBOLS

Re-entry Body Parameters

B - cone half-angle

I - pitch or yaw moment of inertia (slug-ft^2)

I_x - roll moment of inertia (slug-ft^2)

m - mass of vehicle (slugs)

N - loading parameter

Δ_x - static margin of stability (ft)

δ_z - displacement of the c.g. from longitudinal axis (ft)

Aerodynamic Terms

$A_{\alpha,\beta}$ - total angle of attack

A_D - drag parameter

A_L - lift parameter

A_M, K_M - damping moment parameters

C_D - drag coefficient

$\frac{\partial C_{L_\alpha}}{\partial \alpha}$, $C_{L\alpha}$ - lift-curve slope

D - aerodynamic drag

L - aerodynamic lift

α - angle of attack

β - angle of side-slip

Reference Angles and Angular Rates

γ -

θ - orientation angles relating re-entry body

ψ - axes to the inertial reference axes

η } - orientation angles relating the position of the velocity
ϕ_p } - vector with respect to the inertial reference axes

ϕ_o - angle between the reference plane and the horizontal plane

ϕ_{ss} - the steady-state flight-path angle

ω_x -
ω_y - angular rates with respect to an inertial reference; 1/sec about the re-entry body orthogonal axes
ω_z -

Coordinate Axes

X -
Y - reference axes fixed with respect to the earth and considered to be inertial
Z -

x_s -
y_s - re-entry body axes of symmetry
z_s -

x -
y - re-entry body central principal axes
z -

Miscellaneous Terms

g - acceleration of gravity (ft/sec^2)

h - altitude (ft)

k - atmosphere density exponential (a negative number)

V - re-entry-body speed (ft/sec)

R_x - spin rate parameter equal to $\frac{I_x \omega_x}{I}$

I. INTRODUCTION

One of the problems which arises during the re-entry into the atmosphere of a body is the dispersion of the trajectory due to aerodynamic lift forces. Such unwanted forces might result from asymmetries in mass distribution, asymmetries of the body about the longitudinal axis, or other causes. Since very small lift forces can cause errors at impact on the order of several miles,(1) it is desirable to eliminate, or at least to reduce, the effect of these forces upon the re-entry trajectory.

One means of doing this is to reduce the dispersion of the re-entry vehicle by causing the lift vector to precess about the velocity vector. The lift and drag forces exert torques upon the re-entry vehicle, which, if the body is spun about the longitudinal axis, will cause precession about the velocity vector. A high precession rate tends to "average out" the lift vector, thus reducing dispersion.(1)

Although precession of the lift vector about the velocity vector seems to offer a solution to the problem of dispersion due to aerodynamic forces, the dynamic stability of the re-entry body is influenced by the spin rate and thus should be investigated. Furthermore, it may be anticipated that spinning the re-entry body will introduce a unidirectional side force which will cause a lateral displacement of the vehicle. This effect is similar to that which occurs in the case of projectiles fired from rifled guns.

II. DERIVATION OF THE EQUATIONS OF MOTION

Figure 1 defines the coordinate systems and some of the variables pertinent to the analysis of the rotational motion of the spinning re-entry body. The X axis in the reference plane is aligned in the direction of the velocity vector at the beginning of the problem and is considered to be an inertial reference. The orientation angles, γ, θ and ψ, relate the position of the re-entry body orthogonal axes, x_s, y_s and z_s, to the inertial reference axes, X, Y and Z. The angles η and ϕ_p specify the position of the velocity vector with respect to the inertial reference while the angles α and β relate the position of the velocity vector to the re-entry body axes.

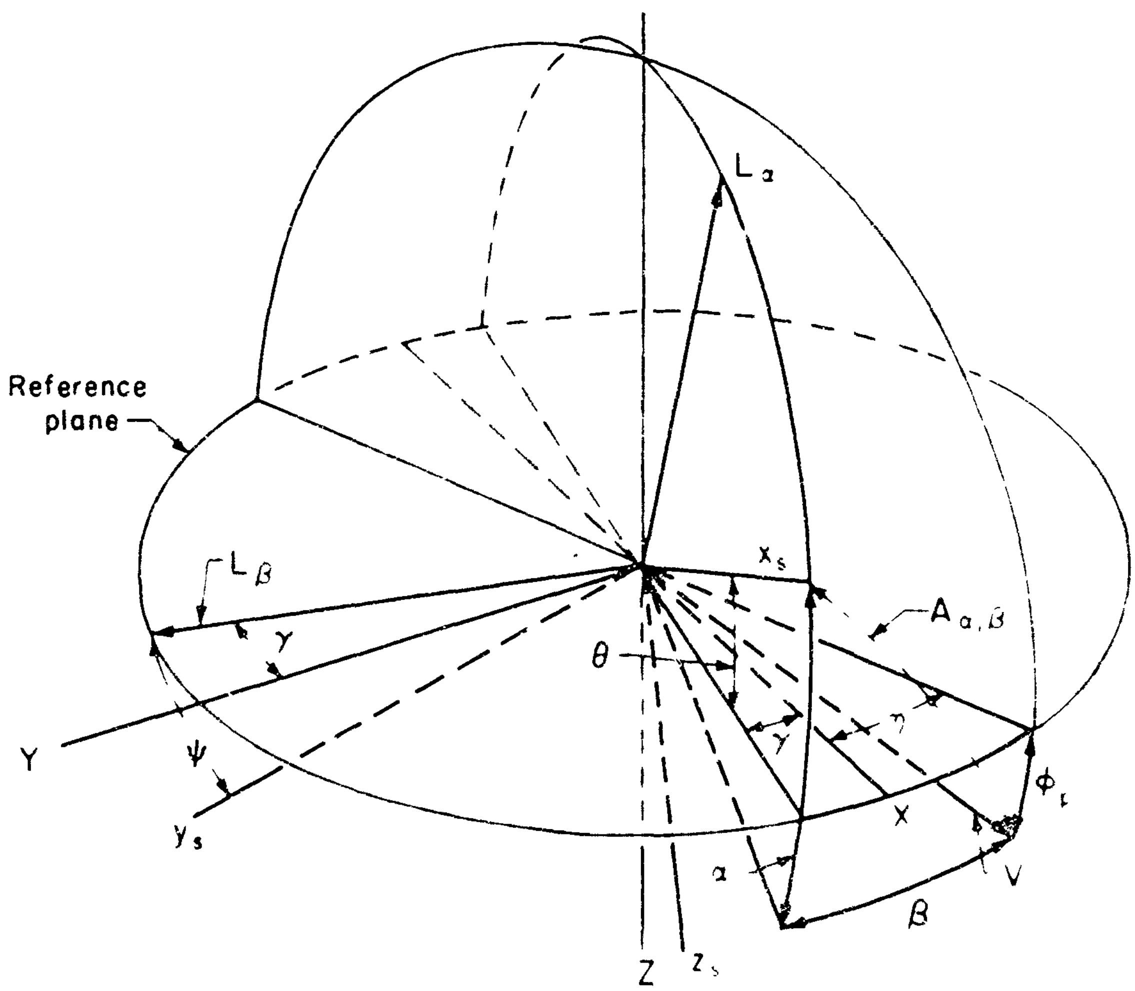

Fig 1 — Re-entry geometry

A complete analysis of the re-entry problem would require the solution of six coupled equations. However, if it is assumed that the three rotational degrees of freedom have a negligible effect upon the translational motion of the center of gravity of the vehicle, then two sets of three equations each may be solved independently. Solutions to the flight path equations must be obtained before solving the rotational equations of motion.

The three translational and the three rotational equations of motion are, respectively:

$$m\dot{V} = mg \sin(\phi_o + \phi_p) - D \tag{1}$$

$$mV\dot{\phi} = mg \cos(\phi_o + \phi_p) - L_\alpha \tag{2}$$

$$mV\dot{\eta} \cos \phi_p = - L_\beta \tag{3}$$

where D is the drag, L_α the lift due to α, L_β the lift due to β, and ϕ_o the angle between the reference and horizontal plane.

$$\dot{\omega}_x + \frac{I_z - I_y}{I_x} \omega_y \omega_z = \frac{M_x}{I_x} \tag{4}$$

$$\dot{\omega}_y + \frac{I_x - I_z}{I_y} \omega_x \omega_z = \frac{M_y}{I_y} \tag{5}$$

$$\dot{\omega}_z + \frac{I_y - I_x}{I_z} \omega_x \omega_y = \frac{M_z}{I_z} \tag{6}$$

where M_x, M_y and M_z are the torques about the x, y and z axes respectively.

The relationship between body angular rates and the rate of change of the orientation angles may be deduced by an examination of Fig. 1:

$$\omega_x = \dot{\psi} - \dot{\gamma} \sin \theta \tag{7}$$

$$\omega_y = \dot{\theta} \cos \psi + \dot{\gamma} \sin \psi \cos \theta \tag{8}$$

$$\omega_z = \dot{\gamma} \cos \psi \cos \theta - \dot{\theta} \sin \psi \tag{9}$$

With the assumption that the angles of attack and side slip are small, the following expressions may be found by an inspection of Fig. 1:

$$\alpha = \theta + \phi_p \tag{10}$$

$$\beta = (\eta + \gamma) \cos \phi_p \tag{11}$$

The form of Eq. (4-6) indicates that the re-entry body axes are principal axes. In Fig. 2, the relationship between the principal axes and the axes of symmetry of the vehicle is indicated. With a nonuniform mass

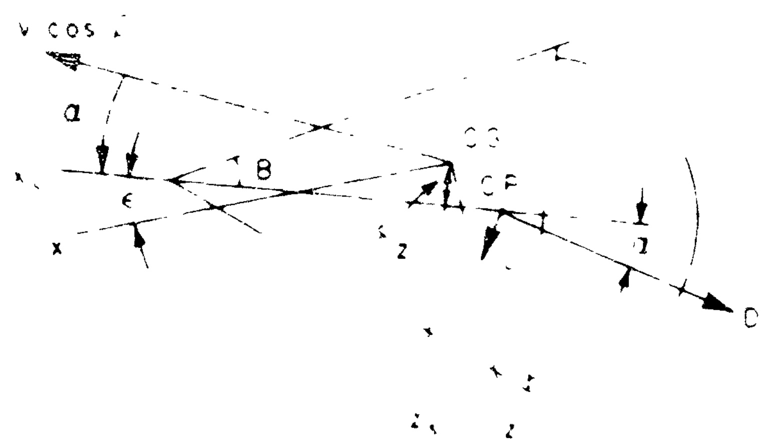

Fig 2 — Re-entry body axes

distribution, the origin of the principal axes, CG, may be displaced from the longitudinal axis, x_s, while the x, z axes may be rotated through an angle ϵ with respect to the x_s, z_s, axes. Moreover, it may be expected that the moments of inertia about the y and z axes would not be equal. However, if δ_z and ϵ are small in magnitude, their products with other small

quantities, such as α and β, may be neglected, and the difference, for a symmetrical body, between I_y and I_z becomes negligible.

The torques about the x, y and z axes are thus

$$\frac{M_x}{I_x} = - F_y \frac{\delta_z}{I_x} \tag{12}$$

$$\frac{M_y}{I_y} = - F_z \frac{\Delta_x}{I_y} + F_x \frac{\delta_z}{I_y} - K_M \omega_y \tag{13}$$

$$\frac{M_z}{I_z} = F_y \frac{\Delta_x}{I_z} - K_M \omega_z \tag{14}$$

where Δ_x is the static margin of stability and δ_z is the displacement of the center of mass of the re-entry body from the longitudinal axis of symmetry. The parameter K_M is related to the stability derivatives, C_{m_q} and $C_{m_{\dot\alpha}}$.

The components of lift and drag along x, y and z are

$$F_x = - D \tag{15}$$

$$F_y \quad L_\beta \cos \psi - L_\alpha \sin \psi - D\alpha \sin \psi + D \beta \cos \psi \tag{16}$$

$$F_z = - L_\beta \sin \psi - L_\alpha \cos \psi - D\alpha \cos \psi - D \beta \sin \psi + D \epsilon \tag{17}$$

Upon assuming an exponential atmosphere, L_α , L_β and D have the following form:

$$L_\alpha \quad A_L e^{kh} V^2 \alpha \tag{18}$$

$$L_\beta = A_L e^{kh} V^2 \beta \tag{19}$$

$$D = A_D e^{kh} V^2 \tag{20}$$

where k is a negative number.

The parameters A_L and A_D may be written

$$A_L = \frac{1}{2} \rho_o A_{ref} \left(\frac{\partial C_L}{\partial \alpha}\right) \tag{21}$$

and

$$A_D = \frac{1}{2} \rho_o C_D A_{ref}. \tag{22}$$

where $\frac{\partial C_L}{\partial \alpha}$ is the lift-curve slope, C_D is the drag coefficient, ρ_o is the sea-level air density and A_{ref} is a reference area.

The particular set of variables utilized to examine re-entry body stability is at the discretion of the investigator. Thus one might use ω_y and ω_z, θ and γ, or α and β. By considering Eq. (2) and (3), in conjunction with Eq. (5-20), expressions may be obtained for α and β.

Combining Eq. (2), (10) and (18) and differentiating yields

$$\ddot{\alpha} + \frac{A_L}{m} V e^{kh} \dot{\alpha} + \alpha \frac{d}{dt}\left[\frac{A_L}{m} V e^{kh}\right] = \ddot{\theta} + \ddot{\phi}_{ss} \tag{23}$$

Utilizing Eq. (5), (6), (8) and (9), we find for $\ddot{\theta}$

$$\ddot{\theta} = \frac{M_y}{I} \cos \psi - \frac{M_z}{I} \sin \psi - \dot{\gamma}\left[\dot{\psi} - \frac{I-I_x}{I} \omega_x\right] \tag{24}$$

Equation (24) may be simplified by the substitution of Eq. (13) and (14), in conjunction with Eq. (8) and (9) and Eq. (15-20). Thus

$$\ddot{\theta} = \frac{\Delta_x}{I} (A_L+A_D) V^2 e^{kh} \alpha - K_M \left[\dot{\alpha} + \frac{A_L}{m} V e^{kh} \alpha - \dot{\phi}_{ss}\right] - \dot{\gamma}\left[\dot{\psi} - \frac{I-I_x}{I} \omega_x\right]$$

$$- (\delta_z + \epsilon \Delta_x) \left(\frac{A_D}{I} V^2 e^{kh}\right) \cos \psi \tag{25}$$

However, from Eq. (3) and (11)

$$\dot{\gamma} = \dot{\beta} + \frac{A_L}{m} V e^{kh} \beta \tag{26}$$

The introduction of Eq. (25) and (26) into Eq. (23) leads to a second-order equation in α, with β and $\dot{\beta}$ coupling terms.

In a manner identical to that indicated above, the corresponding equation in β may be obtained. Thus

$$\ddot{\alpha} + \dot{\alpha}\left[\frac{A_L}{m} V e^{kh} + K_M\right] + \alpha\left[\frac{d}{dt}\left(\frac{A_L}{m} V e^{kh}\right) + K_M \frac{A_L}{m} V e^{kh} + \omega_n^{\,2}\right] \tag{27}$$

$$+ (\dot{\psi}-R_z)\left[\dot{\beta} + \frac{A_L}{m} V e^{kh} \beta\right] = K_M \dot{\phi}_{ss} + \ddot{\phi}_{ss} + \omega_n^{\,2}\left(\frac{A_D}{A_L+A_D}\right)\left(\frac{\delta_z+\epsilon\Delta_x}{\Delta_x}\right) \cos \psi$$

$$\ddot{\beta} + \dot{\beta}\left[\frac{A_L}{m} V e^{kh} + K_M\right] + \beta\left[\frac{d}{dt}\left(\frac{A_L}{m} V e^{kh}\right) + K_M \frac{A_L}{m} V e^{kh} + \omega_n^{\,2}\right] \tag{28}$$

$$- (\dot{\psi}-R_z)\left[\dot{\alpha} + \frac{A_L}{m} V e^{kh}\alpha\right] = - (\dot{\psi}-R_z) \dot{\phi}_{ss} + \omega_n^{\,2} \left(\frac{A_D}{A_L+A_D}\right) \left(\frac{\delta_z+\epsilon\Delta_x}{\Delta_x}\right) \sin \psi$$

where

$$\omega_n^{\,2} = - \frac{\Delta_x}{I} (A_L+A_D) V^2 e^{kh}$$

$$R_z = \frac{I-I_x}{I}\omega_x$$

$$\dot{\phi}_{ss} = \frac{g}{V} \cos \phi_{ss}$$

In deriving Eq. (27) and (28), the following assumptions have been made (1) the angles α, β, ϕ_p, η and γ are 'small angles'; (2) the aerodynamic

parameters A_L and A_D are time-invariant; the moment of inertia about y is equal to the moment of inertia about z; the oscillatory motion of the vehicle has no effect upon V or ϕ_{ss}, the steady-state flight-path angle.

It may be noted that for the case of a non-spinning re-entry body, Eq (27) reduces to the planar form which has been treated by H Julian Allen and others.[2,3,4]

Equations (27) and (28) are coupled, and, for the case in which ω_x is equal to a constant, linear with variable coefficients. The spin rate, ω_x, is constant if the torque about the x axis is zero, since, with I_y equal to I_z, no cross-coupling due to ω_y and ω_z occurs. From Eq (12) and (1?) it can be seen that a displacement of the CG from the axis of symmetry gives rise to a torque, M_x. However, assuming that δ_z is a small quantity, the torques which arise from such an effect may be safely neglected since terms of $\delta_z \alpha$ or $\delta_z \beta$ are second order. Furthermore M_x is an oscillatory function which on the average, approaches zero.

By combining Eq. (7), (10) and (11) we find for ω_x

$$\omega_x = \dot{\psi} - \beta\alpha + \beta \dot{\phi}_p - \beta \dot{\phi}_p \alpha \phi_p + \beta \phi_p \dot{\phi}_p + \gamma\alpha - \gamma \dot{\phi}_p \qquad (29)$$

If we again neglect high-order terms, $\dot{\psi}$ is equal to ω_x.*

In Appendix A, the homogeneous solutions of Eq (27) and (28) are considered, while the forced solutions are treated in Appendix B. Appendix C considers the aerodynamic terms and their relative magnitudes for a typical re-entry body.

A planar re-entry analysis is examined in Appendix D, and the results are compared with those of previous studies.

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*There is one special case of interest when it is necessary to retain at least the second-order terms of Eq. (29). That is when ω_x is equal to zero. Under this condition Eq. (27) and (28) are coupled non-linear equations with time-dependent coefficients.

III. DISCUSSION OF THE RESULTS

The Transient Solutions

The homogeneous solutions of Eq. (27) and (28) as indicated in Appendix A have the following form:

$$\alpha = \frac{1}{\left[\omega_n^2 + \frac{R_x^2}{4}\right]^{1/4}} \left\{ \frac{\dot{\alpha}_o}{R_x^{1/2}\sqrt{2}} \left[e^{-\int_o^t \left[\frac{d}{2} + \frac{R_x d - 2 R_x c}{4\sqrt{\omega_n^2 + \frac{R_x^2}{4}}}\right] dt} \sin\left[Z_1 - Z_1(o)\right] - e^{-\int_o^t \left[\frac{d}{2} + \frac{2 R_x c - R_x d}{4\sqrt{\omega_n^2 + \frac{R_x^2}{4}}}\right] dt} \sin\left[Z_3 - Z_3(o)\right] \right] + \frac{\alpha_o R_x^{1/2}}{\sqrt{2}} e^{-\int_o^t \left[\frac{d}{2} + \frac{2 R_x c - R_x d}{4\sqrt{\omega_n^2 + \frac{R_x^2}{4}}}\right] dt} \cos\left[Z_3 - Z_3(o)\right] \right\} \quad (30)$$

and

$$\beta = \frac{1}{\left[\omega_n^2 + \frac{R_x^2}{4}\right]^{1/4}} \left\{ e^{-\int_o^t \left[\frac{d}{2} + \frac{2 R_x c - R_x d}{4\sqrt{\omega_n^2 + \frac{R_x^2}{4}}}\right] dt} \left[\frac{\dot{\alpha}_o}{R_x^{1/2}\sqrt{2}} \cos\left[Z_3 - Z_3(o)\right] - \frac{\alpha_o R_x^{1/2}}{\sqrt{2}} \sin\left[Z_3 - Z_3(o)\right] \right] - \frac{\dot{\alpha}_o}{R_x^{1/2}\sqrt{2}} e^{-\int_o^t \left[\frac{d}{2} + \frac{R_x d - 2 R_x c}{4\sqrt{\omega_n^2 + \frac{R_x^2}{4}}}\right] dt} \cos\left[Z_1 - Z_1(o)\right] \right\} \quad (31)$$

where $R_x > 0$.

If the forebody of the re-entry vehicle is conical, then the aerodynamic parameters of Eq. (30) and (31) may be written as follows.*

$$d = V e^{kh} \frac{\rho_o g}{2N} \left[(\cot^2 B - 1) + \frac{m^2 g}{36 \pi I N \sin^6 B} + \frac{\Delta_x^2 m}{I} \cot^2 B \right] \qquad (a)$$

$$\omega_n^2 = - \frac{\Delta_x}{I} \frac{\rho_o m g}{2N} \cot^2 B\, V^2 e^{kh} \qquad (b)$$

$$c = \frac{\rho_o g}{2N} (\cot^2 B - 1) V e^{kh} \qquad (c) \quad (32)$$

$$Z_1 = \int \left[\frac{I_x \omega_x}{2I} + \sqrt{- \frac{\Delta_x}{I} \frac{\rho_o m g}{2N} \cot^2 B\, V^2 e^{kh} + \frac{I_x^2 \omega_x^2}{4 I^2}} \right] dt \qquad (d)$$

$$Z_3 = \int \left[- \frac{I_x \omega_x}{2I} + \sqrt{- \frac{\Delta_x}{I} \frac{\rho_o m g}{2N} \cot^2 B\, V^2 e^{kh} + \frac{I_x^2 \omega_x^2}{4 I^2}} \right] dt \qquad (e)$$

As it is noted in Appendix A, the two exponential damping terms are very nearly identical, except when the static margin, Δ_x, approaches zero. Thus for most cases the stability of the transient solution may be deduced by an examination of the term

$$\frac{e^{-\frac{1}{2}\int_o^t d\, dt}}{\left[\omega_n^2 + \frac{R_x^2}{4} \right]^{1/4}}$$

The integral of the damping term may be approximated as follows:

$$-\frac{1}{2}\int_o^t d\, dt = \frac{\rho_o g}{4N} \left[(\cot^2 B - 1) + \frac{m^2 g}{36 \pi I N \sin^6 B} + \frac{\Delta_x^2 m \cot^2 B}{I} \right] \int_{h_o}^{h} \frac{e^{kh}\, dh}{\sin \phi_{ss}} \qquad (33)$$

since

$$\dot{h} = - V \sin \phi_{ss}$$

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*See Appendix C

Down to altitudes of 50,000 feet or so, the steady-state flight-path angle changes very slowly. Thus

$$-\frac{1}{2}\int_{0}^{t} d\,dt \approx \frac{\rho_o g}{4 N k \overline{\sin \phi_{ss}}}\left[(\cot^2 B-1) + \frac{m^2 g}{36\pi I N \sin^6 B} + \frac{\Delta_x^2 m \cot^2 B}{I}\right](e^{kh} - e^{kh_o}) \tag{34}$$

where $\sin \phi_{ss}$ is an average quantity over the altitude interval considered. Since k is negative, the amplitude of the exponential term will decrease with decreasing altitude as long as the quantity within the bracket is positive. For cone half-angles of 45 deg or less, this condition is always fulfilled. Figure 3 is a plot of the normalized exponential exponent as a function of the cone half-angle for a particular re-entry body configuration. An examination of Fig. 3 indicates that for a vehicle with a Δ_x of - 0.5 feet, zero damping occurs for a cone half-angle of 55 deg.

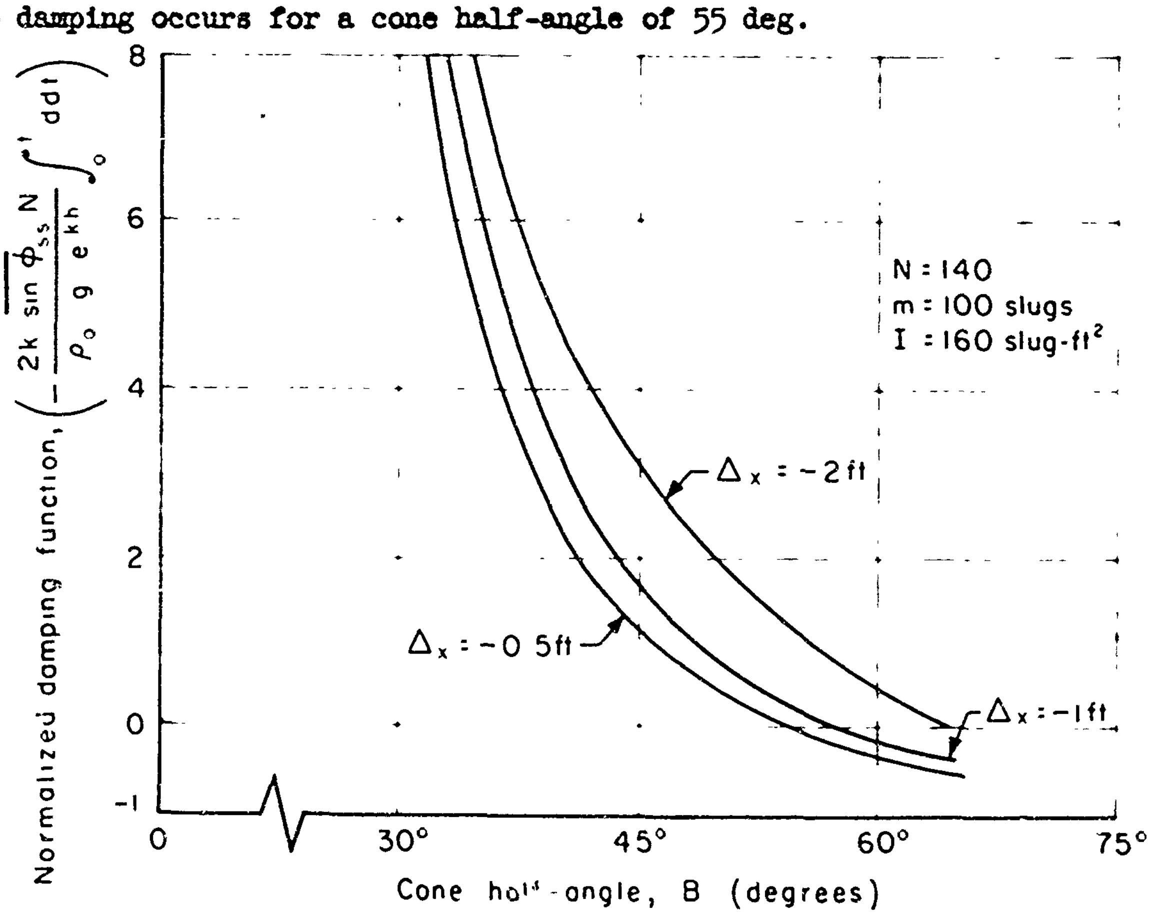

Fig. 3 — Normalized damping function vs re-entry cone half-angle

The remaining factor affecting the amplitude of the oscillatory motion is the term $1/\left[\omega_n^{\,2} + \frac{R_x^{\,2}}{4}\right]^{1/4}$. From Eq. (32b), it can be seen that $\omega_n^{\,2}$ approaches zero at very high altitudes. Thus, since the initial conditions are established when $\omega_n^{\,2}$ is approximately zero, the maximum value attained by the term $1/\left[\omega_n^{\,2} + \frac{R_x^{\,2}}{4}\right]^{1/4}$ is $\sqrt{2}/R_x^{\,1/2}$. As a result of this fact, even with zero damping, the maximum amplitudes of α or β never exceed the initial value of the envelope of the oscillatory motion. Although the magnitude of $\omega_n^{\,2}$ decreases after the peak in dynamic pressure has been passed, it never becomes zero at altitudes below this point. Figure 4, a plot of normalized frequency function versus altitude, illustrates this point.

For the special case in which the static margin is zero, Eq. (30) and (31) simplify to the following forms:

$$\alpha = e^{\frac{A_M}{k \sin \phi_{ss}}\left(e^{kh} - e^{kh_o}\right)} \frac{\dot{\alpha}_o}{R_x} \sin R_x t + \alpha_o\, e^{\frac{A_L}{m\, k \sin \phi_{ss}}\left(e^{kh} - e^{kh_o}\right)} \quad (35)$$

and

$$\beta = -\, e^{\frac{A_M}{k \sin \phi_{ss}}\left(e^{kh} - e^{kh_o}\right)} \frac{\dot{\alpha}_o}{R_x} \cos R_x t + \frac{\dot{\alpha}_o}{R_x}\, e^{\frac{A_L}{m\, k \sin \phi_{ss}}\left(e^{kh} - e^{kh_o}\right)} \quad (36)$$

There are now two distinct exponential terms which influence the stability of the solutions. The exponent containing A_M arises due to C_{m_q}, and thus this term is always convergent. The second exponential, however, is a function of the lift curve slope, C_{L_α}, which, for a conical body with cone half-angles greater than 45 deg, can have a negative value at hypersonic

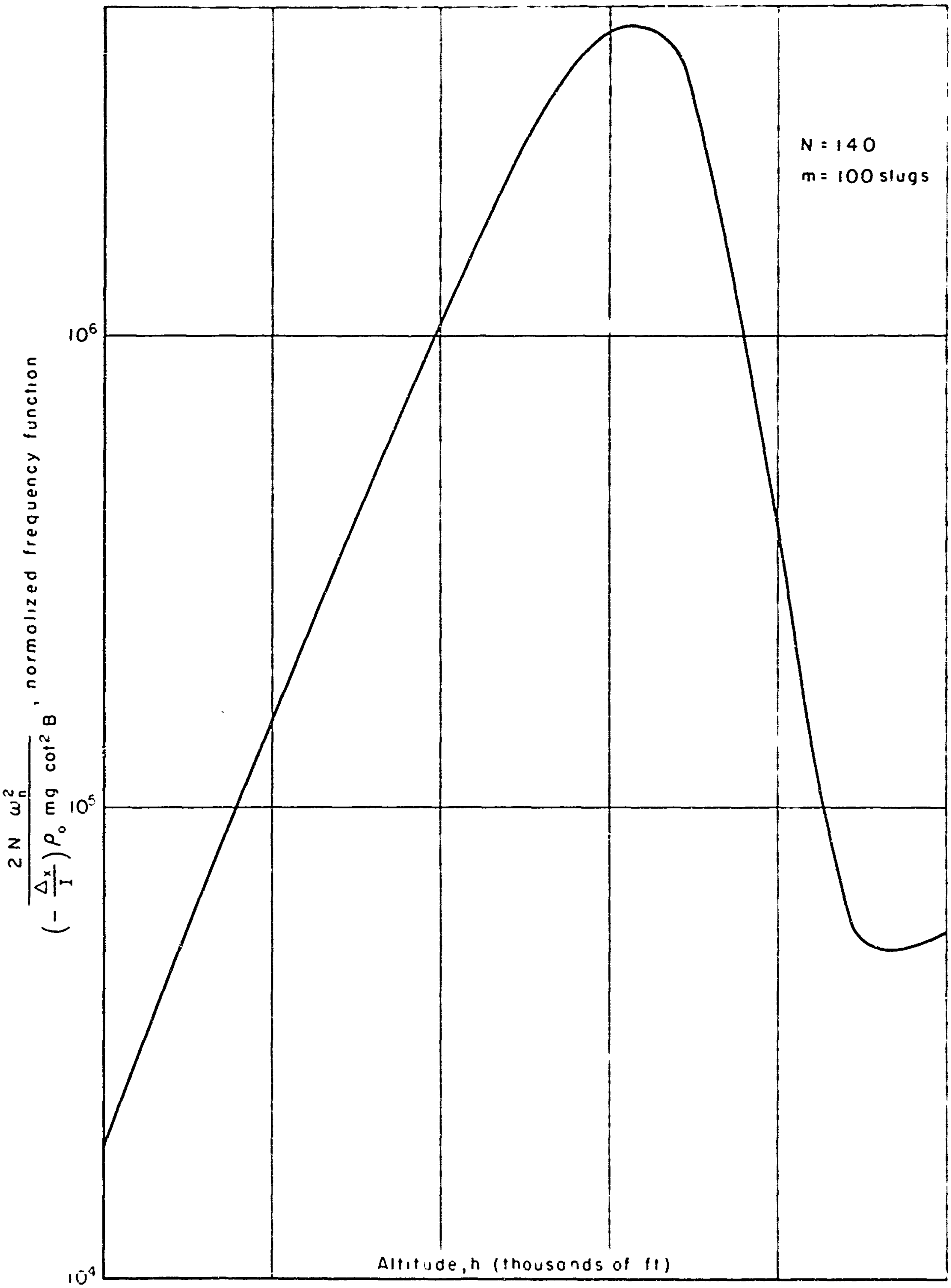

Fig 4 – Normalized frequency function vs altitude

speeds. With a cone half-angle greater than 45 deg, Eq. (35) indicates that α has a divergent, non-oscillatory solution. Of course, if Δ_x is positive, the re-entry body is subjected to a de-stabilizing moment, and divergence will occur.

Before considering the initial conditions of the problem, it is desirable to select a sequence of events in the operation of the attitude control system of the re-entry body. Let us assume that after the last stage of propulsion has ceased, the vehicle is rotated by reaction forces until the longitudinal axis is aligned with the direction the velocity vector will have upon re-entering the atmosphere. After the vehicle is stabilized in this position, an angular acceleration is imparted until a predetermined angular rate about the longitudinal axis is achieved.* At this point the control period is terminated.

The initial conditions that have been selected in the study of the transient behavior of α and β are:

$$\text{at } t = t_o, \text{ or } h = h_o$$

$$\alpha = \alpha_o \; ; \quad \beta = 0$$

$$\dot{\alpha} = \dot{\alpha}_o \; ; \quad \dot{\beta} = 0$$

With an ideal attitude control system, all of the initial conditions would be zero, and only the forced solutions of α and β would exist. For an actual control system, rate and position errors in both α and β should be expected. However, since the primary motion of the longitudinal axis of the re-entry body during the control period is in the plane containing α,

*The dynamics of the spin-up have not been included in this report.

β_o and $\dot{\beta}_o$ should be small in magnitude, and may be neglected.

If the residual body rate is very small compared to the position error, then Eq. (30) and (31) may be approximated as follows:

$$\frac{\alpha}{\alpha_o} = \frac{R_x^{1/2}\, e^{-\int_o^t \left[\frac{d}{2} + \frac{2 R_x c - R_x d}{4\sqrt{\omega_n^2 + \frac{R_x^2}{4}}}\right] dt}}{\sqrt{2}\left[\omega_n^2 + \frac{R_x^2}{4}\right]^{1/4}} \cos\left[Z_3 - Z_3 (o)\right] \tag{37}$$

and

$$\frac{\beta}{\alpha_o} \cong - \frac{R_x^{1/2}\, e^{-\int_o^t \left[\frac{d}{2} + \frac{2 R_x c - R_x d}{4\sqrt{\omega_n^2 + \frac{R_x^2}{4}}}\right] dt}}{\sqrt{2}\left[\omega_n^2 + \frac{R_x^2}{4}\right]^{1/4}} \sin\left[Z_3 - Z_3 (o)\right] \tag{38}$$

where it has been assumed that $R_x >> 0$.

The total transient angle of attack is thus approximately

$$A_{\alpha, \beta} = \frac{\alpha_o R_x^{1/2}\, e^{-\int_o^t \left[\frac{d}{2} + \frac{2 R_x c - R_x d}{4\sqrt{\omega_n^2 + \frac{R_x^2}{4}}}\right] dt}}{\sqrt{2}\left[\omega_n^2 + \frac{R_x^2}{4}\right]^{1/4}} \tag{39}$$

Equation (39) indicates that with an initial $\dot{\alpha}$ of zero, the lift vector precesses about the velocity vector, but nutation does not occur. The rate of precession may be found by differentiating the angle whose tangent is α/β. Thus

$$\dot{\rho} = \frac{1}{\alpha^2 + \beta^2}\left[\beta\dot{\alpha} - \alpha\dot{\beta}\right] \tag{40}$$

Utilizing Eq. (37), (38), and (39), we find for $\dot{\rho}$

$$\dot{\rho} = \dot{Z}_3 = -\frac{R_x}{2} + \sqrt{\omega_n^2 + \frac{R_x^2}{4}} \tag{41}$$

From Eq. (41) it can be seen that as Δ_x goes to zero, $\dot{\rho}$ approaches zero. As Δ_x is increased in a negative sense, $\dot{\rho}$ approaches ω_n, for a fixed value of R_x. The coupling parameter R_x, which is equal to $I_x \omega_{x/I}$, is primarily a function of the spin rate, ω_x. Previous studies have indicated that a spin rate of from 0.5 to 1 rad/sec is required to reduce the effect of steady-state trim angles of attack which arise due to various types of re-entry-body asymmetries. Thus a range of values for R_x of from 0.5 to 2 rad/sec would seem adequate from such considerations. With Δ_x equal to minus 0.5 feet or larger, and with R_x in the range of values indicated above, ω_n is very large compared to R_x. Under such conditions, in the altitude region in which the peak dynamic pressure occurs, ω_n, and thus $\dot{\rho}$, might be on the order of 30 rad/sec. A precession rate of this magnitude is more than sufficient to average out the transient lift vector.

Figures 5, 6 and 7 are plots of $A_{\alpha,\beta}/\alpha_o$ as a function of altitude for various combinations of static margins and nose-cone half-angles. The effect of the spin rate, ω_x, upon $A_{\alpha,\beta}/\alpha_o$ is indicated in Fig. 8.

If the body angular rate at the end of the control period is such that $\dot{\alpha}_o$ is not negligible, the lift vector will nutate as well as precess about the velocity vector. In order to simplify the problem the altitudes considered will be restricted to 200,000 feet and below. With ω_n much larger than R_x, Eq. (30) and (31) become:

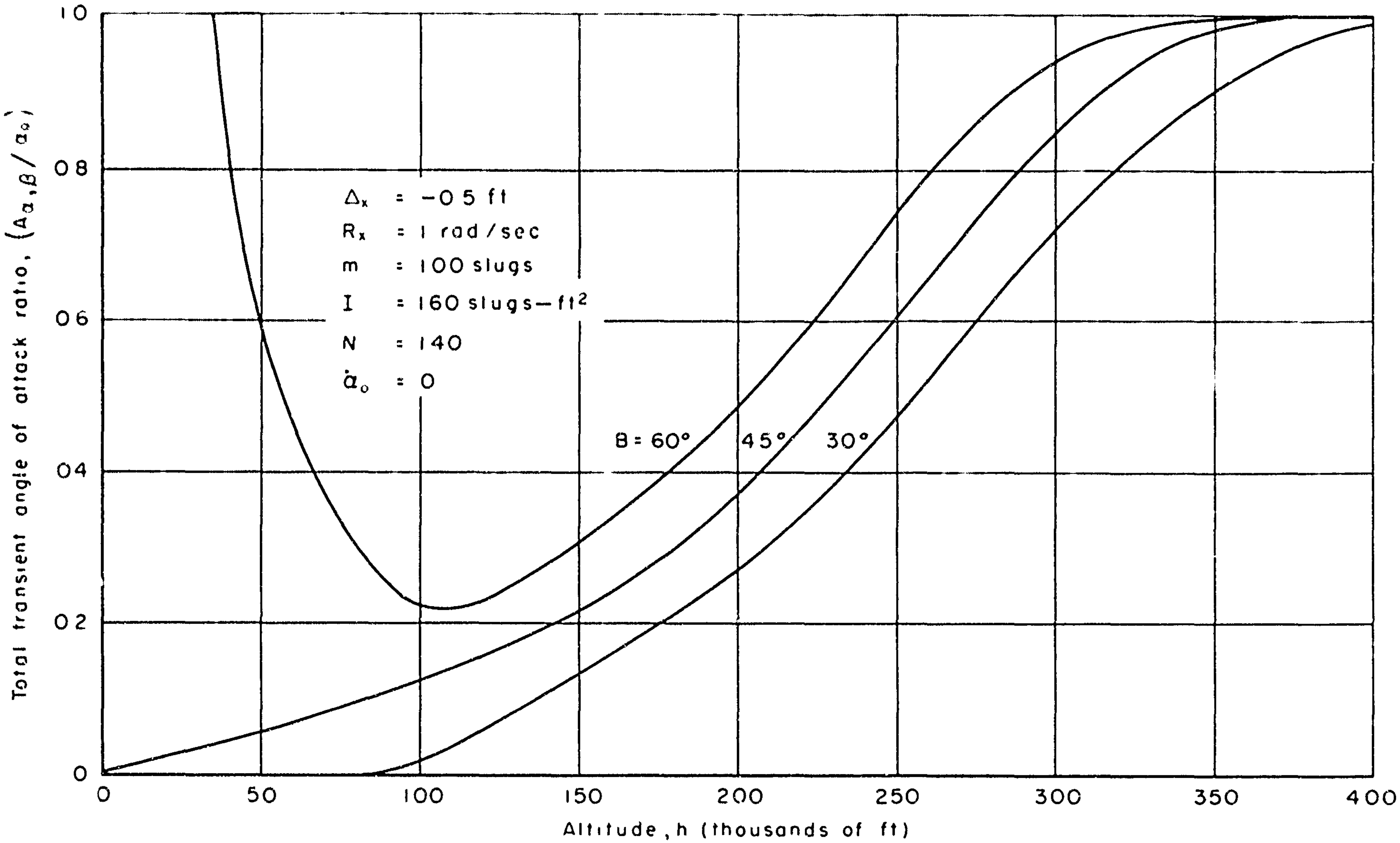

Fig 5 – Total transient angle of attack as a function of altitude
Static margin = −0.5 ft

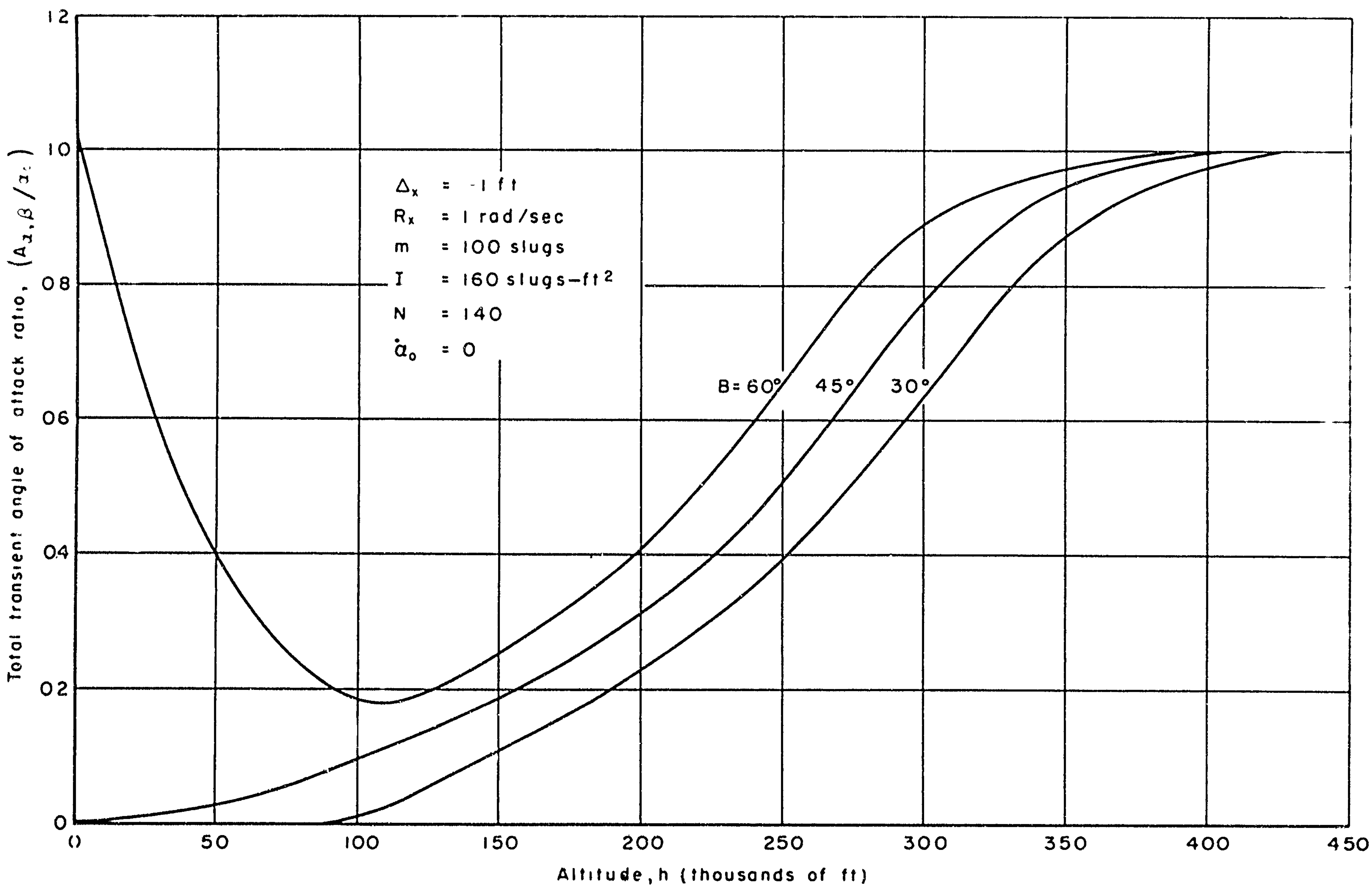

Fig. 6—Total transient angle of attack as a function of altitude

Static margin = −1 ft

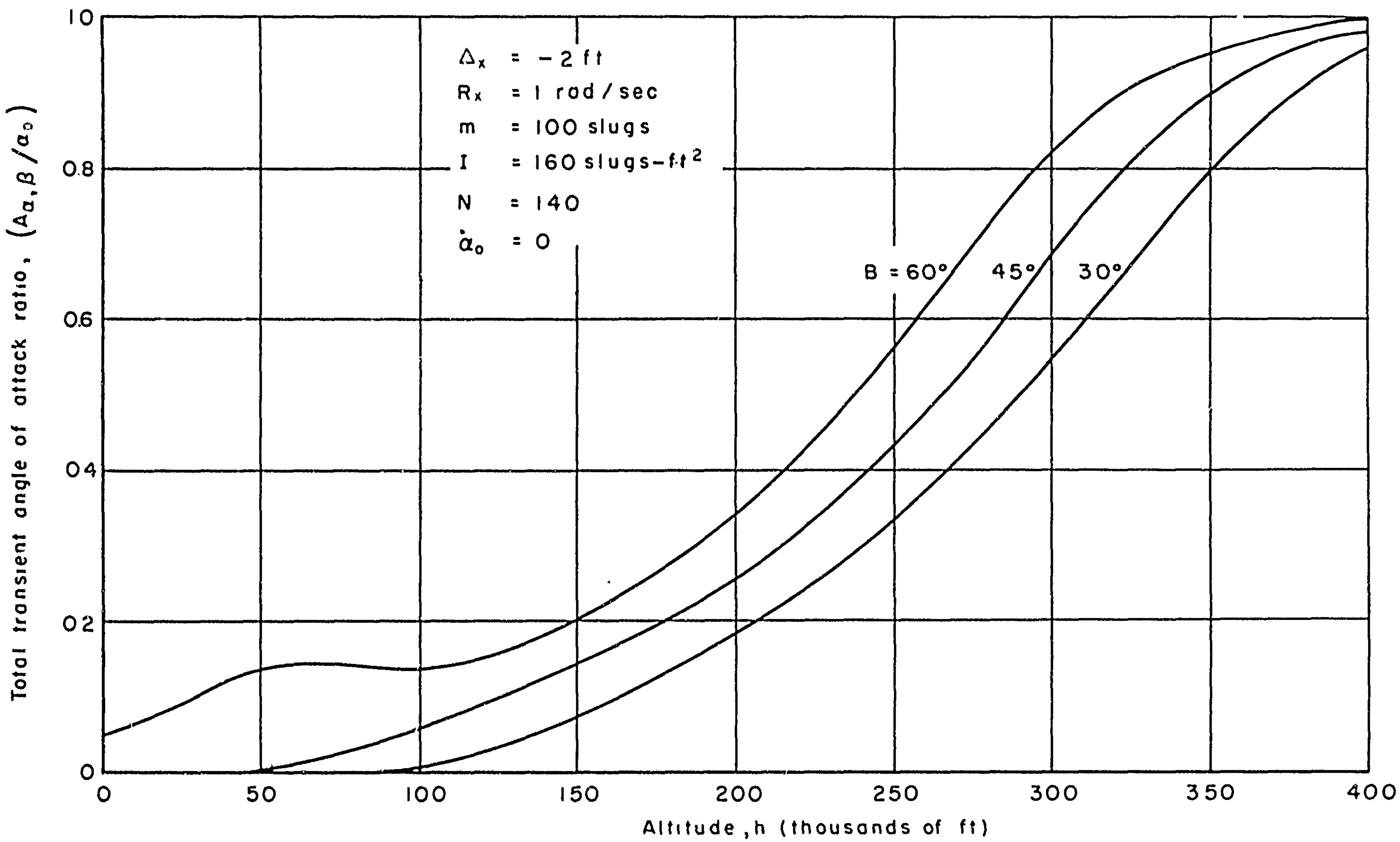

Fig. 7 — Total transient angle of attack as a function of altitude
Static margin = −2 ft

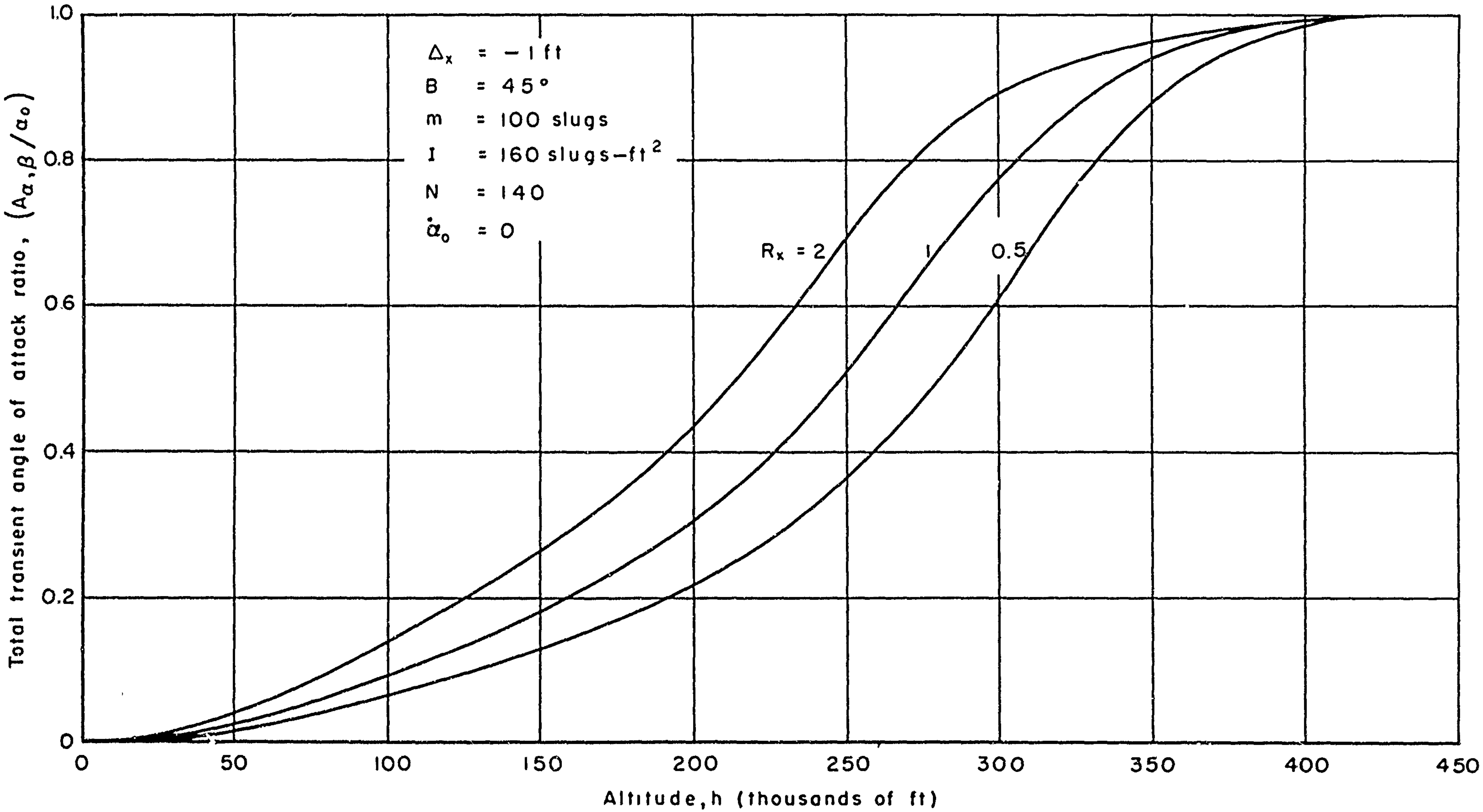

Fig. 8—Total transient angle of attack as a function of altitude

$$\alpha \cong \frac{e^{-\frac{1}{2}\int_0^t d\,dt}}{\left[\omega_n^2 + \frac{R_x^2}{4}\right]^{1/4}} \left\{ \frac{\alpha_o R_x^{1/2}}{\sqrt{2}} \cos\left[Z - \frac{R_x t}{2}\right] + \frac{\sqrt{2}\,\dot{\alpha}_o}{R_x^{1/2}} \cos Z \sin \frac{R_x t}{2} \right\} \quad (42)$$

and

$$\beta \cong \frac{e^{-\frac{1}{2}\int_0^t d\,dt}}{\left[\omega_n^2 + \frac{R_x^2}{4}\right]^{1/4}} \left\{ -\frac{\alpha_o R_x^{1/2}}{\sqrt{2}} \sin\left[Z - \frac{R_x t}{2}\right] + \frac{\sqrt{2}\,\dot{\alpha}_o}{R_x^{1/2}} \sin Z \sin \frac{R_x t}{2} \right\} \quad (43)$$

where $h \leqq 200{,}000$ ft and

$$Z = \int_0^t \omega_n\,dt \cong \frac{2\sqrt{\frac{\Delta_x}{I}(A_L + A_D)}\; e^{\frac{kh}{2}}}{k\sqrt{\sin \phi_{ss}}}$$

The total angle of attack is

$$A_{\alpha,\beta} = \frac{\alpha_o R_x^{1/2}\, e^{-\frac{1}{2}\int_0^t d\,dt}}{\sqrt{2}\left[\omega_n^2 + \frac{R_x^2}{4}\right]^{1/4}} \left[1 + \frac{4\dot{\alpha}_o}{R_x \alpha_o} \sin\frac{R_x t}{2} \cos\frac{R_x t}{2} + \frac{4\dot{\alpha}_o^2}{R_x^2 \alpha_o^2} \sin^2 \frac{R_x t}{2}\right]^{1/2} \quad (44)$$

and the precession rate is

$$\dot{\rho} = \frac{\left(\omega_n - \frac{R_x}{2}\right) - \frac{2\dot{\alpha}_o}{\alpha_o} \sin Z \cos Z - \frac{4\omega_n}{R_x^2}\left(\frac{\dot{\alpha}_o}{\alpha_o}\right)^2 \sin^2\frac{R_x t}{2}}{1 + \frac{4\dot{\alpha}_o}{R_x \alpha_o} \sin\frac{R_x t}{2} \cos\frac{R_x t}{2} + \frac{4}{R_x^2}\left(\frac{\dot{\alpha}_o}{\alpha_o}\right)^2 \sin^2\frac{R_x t}{2}} \quad (45)$$

An examination of Eq. (44) and (45) indicates that nutation becomes important

when α_o has a very large magnitude, or when R_x is small. The exact nature of the precessional motion depends upon the relative magnitudes of α_o, R_x and ω_n. It can be seen from Eq. (45) that $\dot{\rho}$ may actually be negative over portions of the period. When α_o is zero, $\dot{\rho}$ equals minus ω_n. Thus the time history of the motion of the longitudinal axis about the velocity vector is, in general, very complex.

The envelopes of the maximum and minimum amplitudes of the oscillations of $A_{\alpha,\beta}$ are plotted as functions of the altitude in Fig. 9. For the special case in which α_o is finite, $\dot{\alpha}_o$ is zero, and R_x is equal to one, the envelope of $R_x A_{\alpha,\beta}/2\dot{\alpha}_o$ as a function of altitude is given by Fig. 5-7.

In utilizing Fig. 5-9, it should be kept in mind that the small angle approximations that were made in the derivation of the equations of motion in Section II, break down at altitudes on the order of 30,000 to 50,000 ft.

An examination of Fig. 5-9 indicates that increasing the spin rate increases the amplitude of $A_{\alpha,\beta}$. A comparison of Fig. 5 of Section II and Fig. 10 of Appendix D illustrates the differences between the spinning and non-spinning cases. In the limiting case as R_x approaches an infinite value we see from Eq. (30) and (31) that

$$A_{\alpha,\beta} = \alpha_o\, e^{\dfrac{A_L}{m\, k\, \overline{\sin \phi}_{ss}} \left(e^{kh} - e^{kh_o}\right)} \tag{46}$$

Thus for very large values of R_x (or conversely for very low values of ω_n) the stability of the oscillations is dependent only upon the lift curve slope C_{L_α}.

The normal acceleration load acting on the re-entry body is

$$ng = A_{\alpha,\beta}\, \frac{\rho_o\, g}{2N}\, (\cot^2 B - 1)\, V^2\, e^{kh} \tag{47}$$

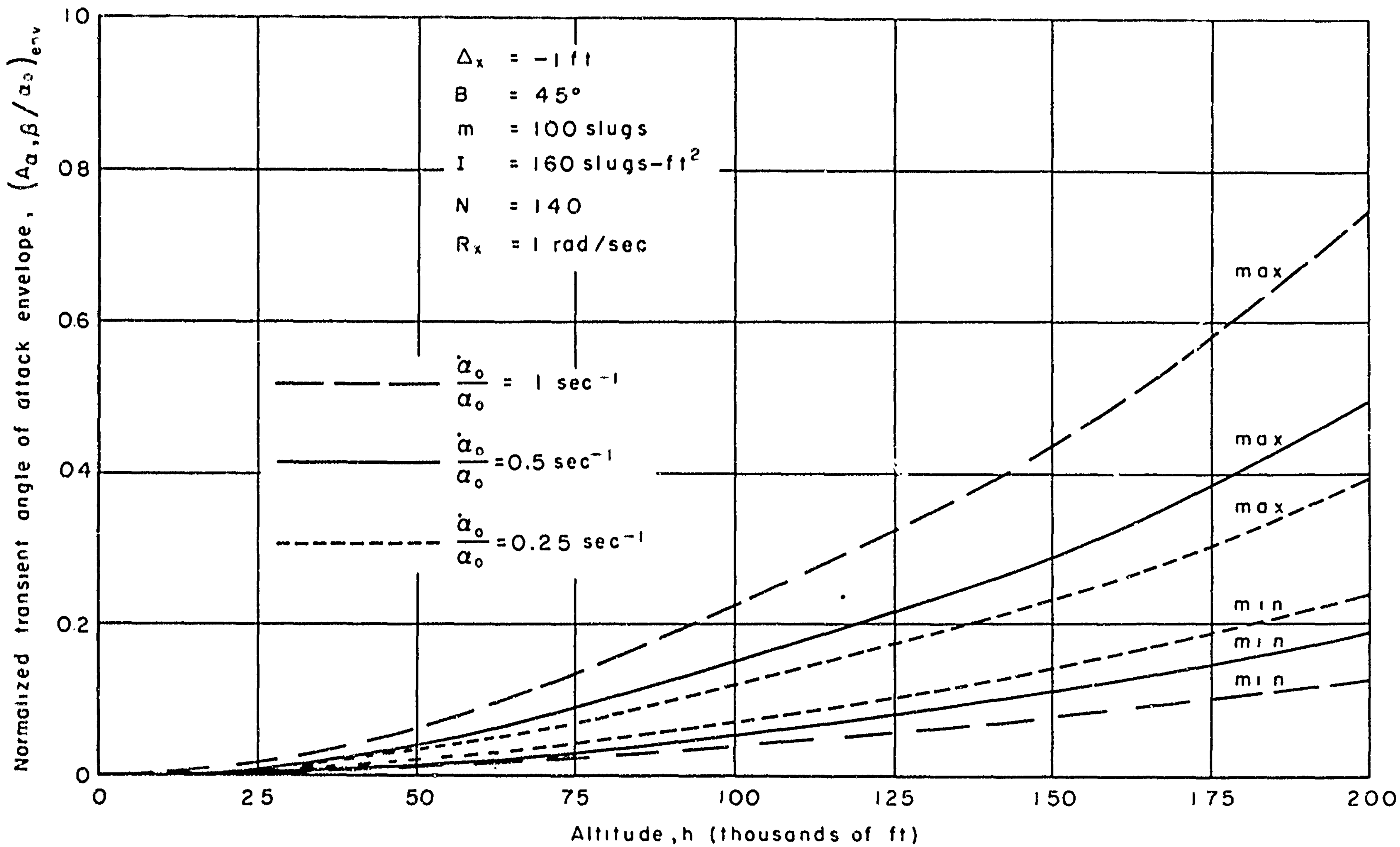

Fig. 9 — Transient angle of attack envelope as a function of altitude

Even for a very large value of the initial angle of attack, such as 20 deg, the maximum value of n for a typical re-entry body varies between zero and five. Zero occurs for B equal to 45 deg.

The Forced Solutions

From Eq. (B-7) and (B-8), we find for the steady-state solutions

$$\alpha_{ss} \cong \frac{\ddot{\phi}_{ss}}{\omega_n^{\,2}} + \frac{K_M \dot{\phi}_{ss}}{\omega_n^{\,2}} + \left(\frac{A_D}{A_L + A_D}\right) \frac{(\delta_z + \epsilon \Delta_x)}{\Delta_x} \cos \omega_x t \tag{48}$$

$$\beta_{ss} \cong - \frac{R_x \dot{\phi}_{ss}}{\omega_n^{\,2}} + \left(\frac{A_D}{A_L + A_D}\right) \frac{(\delta_z + \epsilon \Delta_x)}{\Delta_x} \sin \omega_x t \tag{49}$$

Equations (48) and (49) are good approximations at altitudes of 250,000 ft and less, and if Δ_x is greater than zero. At very high altitudes where the aerodynamic terms of Eq. (27) and (28) approach zero, or when Δ_x is zero, the forced solutions are:

$$\alpha_{ss} = e^{-\int c\, dt} \int e^{\int c\, dt}\, d\, \phi_{ss} \tag{50}$$

$$\beta_{ss} = 0 \tag{51}$$

Let us first consider Eq. (48) and (49). The terms containing δ_z are those that arise due to asymmetries in the mass distribution of the re-entry vehicle. In both equations the terms are multiplied by sinusoidal functions, which depend on the spin rate, ω_x. Reference 1 indicates that with ω_x ranging from 0.5 to 1 rad/sec, the miss distance of the vehicle at impact is approximately one per cent of the value which occurs without spin. The

remaining two terms of Eq. (48) are due to the lag between the angular motion of the velocity vector and the longitudinal axis of the re-entry body.

The first term of Eq. (49) arises due to a coupling between the spinning body and the aerodynamic and gravitational forces acting on the body. It is a unidirectional effect which with a non-zero lift-curve slope will cause the vehicle to depart from the original trajectory plane.[5] Rewriting Eq. (49)

$$\beta_{ss} = \frac{I_x \omega_x \, g \cos \phi_{ss} \, e^{-kh}}{\Delta_x \, (A_L + A_D) \, V^3} + \left(\frac{A_D}{A_L + A_D}\right) \frac{(\delta_z + \epsilon \Delta_x)}{\Delta_x} \sin \omega_x t \tag{52}$$

From Eq. (52) it can be seen that the sign of the unidirectional term depends on the sense of the spin, ω_x, and of the static margin, Δ_x. In the case of a projectile fired from a gun with right-hand rifling, the direction of departure is to the right of the trajectory plane, since Δ_x is positive for a bullet. A re-entry body with a positive spin would move to the left if it is statically stable and has a positive lift-curve slope. The approximate magnitude of the effect may be found as follows:

$$m \ddot{S} \simeq A_L V^2 e^{kh} \beta_{ss} = \frac{I_x \omega_x \dot{\phi}_{ss} A_L}{\Delta_x \, (A_L + A_D)} \tag{53}$$

Integrating Eq. 53 yields the rate of departure from the original trajectory plane. Thus

$$\dot{S} = \frac{I_x \omega_x A_L}{m \, \Delta_x \, (A_L + A_D)} (\phi_{ss} - \phi_o) \tag{54}$$

The lateral displacement, as a function of time is:

$$S = \frac{I_x \omega_x A_L}{m \Delta_x (A_L + A_D)} \int_0^t (\phi_{ss} - \phi_o)\, dt \qquad (55)$$

Initially, ϕ_o is 16 deg, while at impact ϕ_{ss} is 90 deg. An order of magnitude value of S may be obtained by approximating $(\phi_{ss} - \phi_o)$, since the integral of Eq. (55) may be expressed as follows:

$$\int_0^t (\phi_{ss} - \phi_o)\, dt = t\, (\overline{\phi_{ss} - \phi_o}) \qquad (56)$$

For a typical re-entry body, with N equal to 140, a mass of 100 slugs, and I equal to 160 slug-ft^2, we find for S

$$S = \frac{R_x (1 - \tan^2 B)}{\Delta_x} (77.5) \qquad (57)$$

With a 60 deg cone half-angle, an R_x of 2 and a Δ_x of minus 0.5 ft, S is a positive 620 ft. It should be remembered that Eq. (57) is not valid when Δ_x approaches zero. However, from an examination of Eq. (B-5) and (B-6) it is evident that Δ_x may become very small before the approximation is no longer applicable. Thus the unidirectional sideslip term can lead to large values of lateral dispersion.

When Δ_x is zero, Eq. (50) and (51) represent exact solutions of the steady-state equations of motion. The only forced disturbance with a static margin of zero is in α_{ss}.

From a comparison of Eq. (49) and (51), it appears that as Δ_x becomes small the magnitude of the unidirectional term increases until a maximum is reached. As Δ_x continues to decrease, β_{ss} also decreases until it is zero with Δ_x equal to zero.

IV. CONCLUSIONS

From the analysis of the transient and forced solutions for α and β, it is possible to draw certain conclusions concerning the dynamics of the re-entry vehicle's angular motion:

1. Unless the static margin is positive the envelope of the oscillations of a re-entry vehicle whose forebody is conical will never diverge if the cone half-angle, B, is 45 deg or less. For large negative values of the static margin, B may be larger than 45 deg and the envelope will still be convergent (see Fig. 7). Unless the parameter R_x is very large, or the static margin is very small, spinning the vehicle has no effect upon the dynamic stability of the body.
2. The amplitude of the oscillations increases as R_x increases. At altitudes on the order of 200,000 ft or less, the frequency of the oscillations is completely determined by the magnitude of the static moment unless R_x is very large or Δ_x is very small.
3. The effect of the forced or steady-state solutions of α and β due to mass asymmetries is reduced by increasing ω_x, since spin tends to 'average out' the lift vector. However, the unidirectional sideslip term increases as R_x increases.

A value of R_x of from 0.5 to 2 rad/sec is adequate from the viewpoint of averaging out re-entry-body asymmetries. With R_x in the above range, the unidirectional sideslip term is of little importance unless Δ_x, the static margin, is very small. However, the problem of dispersion becomes relatively unimportant if the re-entry vehicle is designed so that the

lift-curve slope, C_{L_α}, is zero.

Spin about the longitudinal axis is desirable, even with C_{L_α} equal to zero, in order to prevent a large initial value of α from arising due to small residual rate errors at the end of the control period.

Under all circumstances it is necessary that Δ_x be less than zero. From Eq. (37) and (38) it can be seen that a Δ_x of zero does not imply divergent oscillations. However, the damping of the motion is very weak, and as a consequence relatively large values of α or β might occur in the lower atmosphere. This would be undesirable due to the problems associated with aerodynamic heating.

APPENDIX A

The Homogeneous Solutions of the Equations of Motion

With the assumption that the rotational motion of the re-entry body does not affect the time history of V, h and ϕ, these variables may be determined from the three flight-path equations of motion. Equations (27) and (28) are thus coupled linear equations, the coefficients of which are known functions of time.

Rewriting Eq. (27) and (28) in the homogeneous form

$$\ddot{\alpha} + d\dot{\alpha} + \Omega^2\alpha + R_x\left[\dot{\beta} + c\,\beta\right] = 0 \qquad \text{(A-1)}$$

$$\ddot{\beta} + d\,\dot{\beta} + \Omega^2\beta - R_x\left[\dot{\alpha} + c\,\alpha\right] = 0 \qquad \text{(A-2)}$$

where

$$d = \frac{A_L}{m} V e^{kh} + K_M$$

$$\Omega^2 = \omega_n^2 + \frac{d}{dt}\left(\frac{A_L}{m} V e^{kh}\right) + K_M \frac{A_L}{m} V e^{kh}$$

$$R_x = \frac{I_x \omega_x}{I} = \dot{\psi} - R_z$$

$$c = \frac{A_L}{m} V e^{kh}$$

As an example of the method of solution that is to be employed, the planar oscillation of the re-entry body without spin is of interest. With $\psi = \dot{\psi} = 0$

$$\ddot{\alpha} + d\,\dot{\alpha} + \Omega^2\alpha = 0 \qquad \text{(A-3)}$$

Let us assume as the solution of Eq. (A-3) the following expression:

$$\alpha = A e^{\int \lambda dt} \; ; \; A = \text{const.} \tag{A-4}$$

Differentiating Eq. (A-4) and substituting into Eq. (A-3) yields

$$\dot{\lambda} + \lambda^2 + d\lambda + \Omega^2 = 0 \tag{A-5}$$

If the coefficients d and Ω^2 were constants, then two roots which are particular solutions of Eq. (A-5) are

$$\lambda = -\frac{d}{2} \pm \frac{1}{2} \sqrt{d^2 - 4\Omega^2} \tag{A-6}$$

However, since the coefficients are time variables, λ equal to a constant does not satisfy Eq. (A-5).

At this point, let us make the following transformation:

$$\lambda = \sigma + j\,\omega \tag{A-7}$$

Substituting Eq. (A-7) into Eq. (A-5) and simplifying yields

$$\sigma^2 + 2\,j\,\sigma\omega - \omega^2 + \dot{\sigma} + j\,\dot{\omega} + d\,\sigma + j\,d\omega + \Omega^2 = 0 \tag{A-8}$$

Separating the real and imaginary parts of Eq. (A-8)

$$\sigma = -\frac{d}{2} - \frac{\dot{\omega}}{2\omega} \tag{A-9}$$

and

$$\omega^2 = \Omega^2 + \sigma^2 + d\,\sigma + \dot{\sigma} \tag{A-10}$$

From Eq. (A-9) it can be seen that even if there were no aerodynamic damping of the re-entry body, i.e., d equal to zero, the variable coefficients affect the amplitude of the oscillations through the change and the rate of change

of the damped natural frequency, ω. Combining Eq. (A-9) and (A-10) yields for ω^2

$$\omega^2 = \Omega^2 - \frac{d^2}{4} - \frac{\dot{d}}{2} + \frac{3}{4}\left(\frac{\dot{\omega}}{\omega}\right)^2 - \frac{1}{2}\frac{\ddot{\omega}}{\omega} \tag{A-11}$$

Since the real part of λ as given by Eq. (A-9) will integrate directly, the solution of Eq. (A-3) is

$$\alpha = \frac{e^{-1/2\int d\,dt}}{\omega^{1/2}}\left\{c_1 e^{j\int \omega dt} + c_2 e^{-j\int \omega dt}\right\} \tag{A-12}$$

The damped natural frequency, ω, remains to be determined. Before any solution for ω may be obtained from Eq. (A-11), it is necessary to know Ω and d as functions of time.

Upon an examination of the relative magnitudes of the terms of Eq. (A-11), it is found that, at least for conical re-entry bodies

$$\left|\Omega^2\right| \gg \left|\frac{d^2}{4} + \frac{\dot{d}}{2}\right| \tag{A-13}$$

That is, the rate damping of such a vehicle is so small that the damped and undamped natural frequencies are for all practical purposes identical.* Furthermore, from the definition of Ω^2 on page 31 it is evident that

$$\Omega^2 \cong \omega_n^2 \tag{A-14}$$

Thus as a first approximation to the solution of Eq. (A-11)

$$\omega = \pm\omega_n \tag{A-15}$$

Equation (A-15) represents the exact solution of Eq. (A-11) (still neglecting the extremely small d^2 and $\dot{d}$ terms) if

$$\Omega_1^2 = \omega_n^2 - \frac{3}{4}\left(\frac{\dot{\omega}_n}{\omega_n}\right)^2 + \frac{1}{2}\frac{\ddot{\omega}_n}{\omega_n} \tag{A-16}$$

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*See the discussion of the damping terms in Ref. 2. Also, see Appendix C.

where the subscript '1' indicates the first approximation of the coefficient Ω^2. If for the particular conditions under consideration the derivative terms of Eq. (A-16) are not negligible compared to ω_n^2, then a second iteration must be utilized. Thus

$$\omega = \pm\sqrt{\omega_n^2 + \frac{3}{4}\left(\frac{\dot{\omega}_n}{\omega_n}\right)^2 - \frac{1}{2}\frac{\ddot{\omega}_n}{\omega_n}} \tag{A-17}$$

and

$$\Omega_2^2 = \omega_n^2 + \frac{3}{4}\left[\left(\frac{\dot{\omega}_n}{\omega_n}\right)^2 - \left(\frac{\dot{\omega}}{\omega}\right)^2\right] - \frac{1}{2}\left[\frac{\ddot{\omega}_n}{\omega_n} - \frac{\ddot{\omega}}{\omega}\right] \tag{A-18}$$

The iteration process may be continued until Ω_I^2 is as close to ω_n^2 as is desired. For the problem at hand, the following expression has been found to be quite adequate:

$$\omega = \pm\sqrt{\omega_n^2 + \frac{1}{16}k^2 V^2 \sin^2 \phi} \tag{A-19}$$

The ω_n^2 term under the radical of Eq. (A-19) is much larger (on the order of a factor of 100 at 250,000 ft altitude) than the remaining term except at the extreme altitudes at which the initial conditions of the problem must be established. Equation (A-12) thus becomes

$$\alpha \cong \alpha_o \frac{\left(\frac{1}{16}\right)^{1/4}\sqrt{-k V_o \sin \phi_o}\; e^{-\frac{1}{2}\int_o^t d\, dt}}{\left[\omega_n^2 + \frac{1}{16}k^2 V^2 \sin^2 \phi\right]^{1/4}} \cos(Z - Z_o) \tag{A-20}$$

where

$$Z = +\int \omega\, dt$$

In obtaining Eq. (A-20), it has been assumed that re-entry starts at an altitude of approximately 500,000 ft. At this altitude there is an angle, α_0, between the velocity vector and the longitudinal axis of the vehicle. The rate of change of α is initially zero.

The planar case is considered in greater detail in Appendix D.

Let us return to Eq. (A-1) and (A-2) and consider the coupled case in which the re-entry body is spinning about its longitudinal axis. If the coefficients of Eq. (A-1) and (A-2) were constant, solutions might be obtained by any one of several methods. Perhaps the most convenient approach is to assume an exponential solution for both α and β and then derive the secular equation (e.g., see Ref. 6). The secular equation may then be solved for the required roots. An analogous procedure may be used for this particular problem, even though the coefficients are functions of time.* Let us assume solutions for α and β of the following form:

$$\alpha = A\, e^{\int \lambda dt} \tag{A-21}$$

$$\beta = B\, e^{\int \lambda dt} \tag{A-22}$$

where A and B are arbitrary constants.

Substituting α and β and their derivatives, as obtained from Eq. (A-21) and (A-22), into Eq. (A-1) and (A-2) yields

$$A\,(\dot{\lambda} + \lambda^2 + d\lambda + \Omega^2) + B\,R_x\,(\lambda + c) = 0 \tag{A-23}$$

$$-A\,R_x\,(\lambda + c) + B\,(\dot{\lambda} + \lambda^2 + d\lambda - \Omega^2) = 0 \tag{A-24}$$

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*This is true since the problem represents the so-called 'degenerate case.' That is, with zero coupling the α and β equations would have identical solutions.

with A and B known constants, a non-trivial solution exists only if the determinant of their coefficients equals zero, thus

$$(\dot{\lambda} + \lambda^2 + d\lambda + \Omega^2)^2 + R_x^2\,(\lambda + c)^2 = 0 \tag{A-25}$$

Again taking λ to be a complex quantity

$$\lambda = \sigma + j\omega \tag{A-26}$$

Substituting Eq. (A-26) into Eq. (A-25), and separating the real and imaginary parts yields

$$\sigma = -\frac{d\,\omega}{2\omega \mp R_x} - \frac{\dot{\omega}}{2\omega \mp R_x} \pm \frac{R_x\,c}{2\omega \mp R_x} \tag{A-27}$$

and

$$\omega^2 \mp R_x\omega = \Omega^2 \pm \frac{R_x\,c\,d}{2\omega \mp R_x} - \frac{d^2\,\omega}{2\omega \mp R_x} - 2\,\frac{d\,\dot{\omega}}{2\omega \mp R_x} + \frac{d^2\,\omega^2}{(2\omega \mp R_x)^2}$$

$$- \frac{\dot{d}\,\omega}{2\omega \mp R_x} + \frac{4\,d\,\omega\,\dot{\omega}}{(2\omega \mp R_x)^2} \mp \frac{2\,R_x\,c\,d\omega}{(2\omega \mp R_x)^2} \mp \frac{4\,R_x\,c\,\dot{\omega}}{(2\omega \mp R_x)^2}$$

$$+ \frac{R_x^2\,c^2}{(2\omega \mp R_x)^2} \pm \frac{R_x\,\dot{c}}{2\omega \mp R_x} + \frac{3\,\dot{\omega}^2}{(2\omega \mp R_x)^2} - \frac{\ddot{\omega}}{2\omega \mp R_x} \tag{A-28}$$

If we examine the magnitude of the various terms of Eq. (A-28), it is apparent that those containing c and d are very small compared to Ω^2.

As a first approximation of ω we find

$$\omega = \pm \frac{R_x}{2} \pm \sqrt{\omega_n^2 + \frac{R_x^2}{4}} \tag{A-29}$$

An examination of the ω derivative terms of Eq. (A-28) reveals that Eq. (A-29) represents a satisfactory approximation over an altitude range of 500,000 ft to sea level. Thus the four frequencies of oscillation are:

$$\omega_1 = \frac{R_x}{2} + \sqrt{\omega_n^2 + \frac{R_x^2}{4}} \qquad (a) \qquad\qquad (A\text{-}30)$$

$$\omega_2 = \frac{R_x}{2} - \sqrt{\omega_n^2 + \frac{R_x^2}{4}} \qquad (b)$$

$$\omega_3 = -\frac{R_x}{2} + \sqrt{\omega_n^2 + \frac{R_x^2}{4}} \qquad (c)$$

$$\omega_4 = -\frac{R_x}{2} - \sqrt{\omega_n^2 + \frac{R_x^2}{4}} \qquad (d)$$

The corresponding σ's are

$$\sigma_1 = -\frac{d}{2} - \frac{R_x d}{4\sqrt{\omega_n^2 + \frac{R_x^2}{4}}} + \frac{R_x c}{2\sqrt{\omega_n^2 + \frac{R_x^2}{4}}} - \frac{1}{2}\,\frac{\frac{d}{dt}\sqrt{\omega_n^2 + \frac{R_x^2}{4}}}{\sqrt{\omega_n^2 + \frac{R_x^2}{4}}} \qquad (a) \qquad (A\text{-}31)$$

$$\sigma_2 = -\frac{d}{2} + \frac{R_x d}{4\sqrt{\omega_n^2 + \frac{R_x^2}{4}}} - \frac{R_x c}{2\sqrt{\omega_n^2 + \frac{R_x^2}{4}}} - \frac{1}{2}\,\frac{\frac{d}{dt}\sqrt{\omega_n^2 + \frac{R_x^2}{4}}}{\sqrt{\omega_n^2 + \frac{R_x^2}{4}}} \qquad (b)$$

$$\sigma_3 = \sigma_2 \qquad (c)$$

$$\sigma_4 = \sigma_1 \qquad (d)$$

Equation (A-31), (a) through (d), reveals that the two sets of roots are not identically damped. However, an investigation of relative magnitudes indicates that the middle two terms of Eq. (A-31), (a) and (b), may be neglected in comparison to the remaining terms in the range of altitudes from 250,000 ft to sea level. Thus the solutions of Eq. (A-1) and (A-2) have the following form:

$$\alpha = \frac{e^{-\frac{1}{2}\int d\,dt}}{\left[\omega_n^2 + \frac{R_x^2}{4}\right]^{1/4}} \left\{ A_1 \cos Z_1 + A_2 \sin Z_1 + A_3 \cos Z_3 + A_4 \sin Z_3 \right\} \qquad \text{(A-32)}$$

$$\beta = \frac{e^{-\frac{1}{2}\int d\,dt}}{\left[\omega_n^2 + \frac{R_x^2}{4}\right]^{1/4}} \left\{ B_1 \cos Z_1 + B_2 \sin Z_1 + B_3 \cos Z_3 + B_4 \sin Z_3 \right\} \qquad \text{(A-33)}$$

where

$$Z_1 = \int \omega_1 \, dt$$

$$Z_3 = \int \omega_3 \, dt$$

Of the eight constants of integration that appear in Eq. (A-32) and (A-33), only four are independent, since only four total initial conditions may be specified. The choice of initial conditions is dependent upon the assumed performance of the attitude control system which positions the vehicle for re-entry. Thus if at the end of the control period, the body has both an error in attitude, and a residual body angular rate, then the following set of initial conditions are reasonable:

$$\alpha_o = \alpha_o$$

$$\dot{\alpha}_o = \dot{\alpha}_o \qquad \text{(A-34)}$$

$$\beta_o = 0$$

$$\dot{\beta}_o = 0$$

If such a set of initial conditions is selected, the following constants of integration may be obtained from Eq. (A-32) and (A-33) when the initial altitude is such that the influence of the aerodynamic terms is negligible.

$$A_1 \cong 0 \qquad \text{(a)}$$

$$A_2 \cong \frac{\dot{\alpha}_o}{R_x^{1/2}\sqrt{2}} \qquad \text{(b)} \qquad \text{(A-35)}$$

$$A_3 \cong \frac{\alpha_o R_x^{1/2}}{\sqrt{2}} \qquad \text{(c)}$$

$$A_4 \cong - \frac{\alpha_o}{\sqrt{2}\, R_x^{1/2}} \qquad \text{(d)}$$

and

$$B_1 \cong - \frac{\dot{\alpha}_o}{R_x^{1/2}\sqrt{2}} \qquad \text{(a)}$$

$$B_2 \cong 0 \qquad \text{(b)} \qquad \text{(A-36)}$$

$$B_3 \cong + \frac{\dot{\alpha}_o}{R_x^{1/2}\sqrt{2}} \qquad \text{(c)}$$

$$B_4 \sim - \frac{\alpha_o R_x^{1/2}}{\sqrt{2}} \qquad \text{(d)}$$

For the assumed conditions, the solutions of Eq. (A-32) and (A-33) may be written as follows:

$$\alpha = \frac{e^{-\frac{1}{2}\int_0^t d\,dt}}{\left[\omega_n^2 + \frac{R_x^2}{4}\right]^{1/4}} \left\{ \frac{\alpha_o}{R_x^{1/2}\sqrt{2}} \sin\left[Z_1 - Z_1(o)\right] + \frac{\alpha_o R_x^{1/2}}{\sqrt{2}} \cos\left[Z_3 - Z_3(o)\right] - \frac{\alpha_o}{R_x^{1/2}\sqrt{2}} \sin\left[Z_3 - Z_3(o)\right] \right\} \quad \text{(A-37)}$$

and

$$\beta = \frac{e^{-\frac{1}{2}\int_0^t d\,dt}}{\left[\omega_n^2 + \frac{R_x^2}{4}\right]^{1/4}} \left\{ \frac{\dot{\alpha}_o}{R_x^{1/2}\sqrt{2}} \left[\cos\left[Z_3 - Z_3(o)\right] - \cos\left[Z_1 - Z_1(o)\right]\right] - \frac{\alpha_o R_x^{1/2}}{\sqrt{2}} \sin\left[Z_3 - Z_3(o)\right] \right\} \quad \text{(A-38)}$$

In obtaining the approximate solutions for α and β as given by Eq. (A-37) and (A-38), it has been assumed that the term containing the static margin, ω_n^2, is predominant from approximately 250,000 ft to sea level. For the special case in which the static margin is zero, the four roots are:

$$\begin{aligned} \lambda_1 &= -K_M + j R_x & \text{(a)} \\ \lambda_2 = \lambda_3 &= -c & \text{(b)} \\ \lambda_4 &= -K_M - j R_x & \text{(c)} \end{aligned} \quad \text{(A-39)}$$

With the initial conditions given by Eq. (A-34) the six constants of integration are:

$$A_1 = 0 \qquad \text{(a)}$$

$$A_2 = \frac{\dot{\alpha}_o}{R_x} \qquad \text{(b)} \qquad \text{(A-40)}$$

$$A_3 = \alpha_o \qquad \text{(c)}$$

and

$$B_1 = -\frac{\dot{\alpha}_o}{R_x} \qquad \text{(a)}$$

$$B_2 = 0 \qquad \text{(b)} \qquad \text{(A-41)}$$

$$B_3 = \frac{\dot{\alpha}_o}{R_x} \qquad \text{(c)}$$

For the case of zero static margin, Eq. (A-32) and (A-33) have solutions of the following form:

$$\alpha = e^{-\int K_M\, dt} \frac{\dot{\alpha}_o}{R_x} \sin R_x t + \alpha_o e^{-\int c\, dt} \qquad \text{(A-42)}$$

and

$$\beta = -e^{-\int K_M\, dt} \frac{\dot{\alpha}_o}{R_x} \cos R_x t + \frac{\dot{\alpha}_o}{R_x} e^{-\int c\, dt} \qquad \text{(A-43)}$$

APPENDIX B

The Forced Solutions of the Equations of Motion

From Section II, the coupled equations in α and β are:

$$\ddot{\alpha} + d\,\dot{\alpha} + \Omega^2 \alpha + R_x\,(\dot{\beta} + c\,\beta) = \ddot{\phi}_{ss} + K_M\,\dot{\phi}_{ss} + \omega_n^{\,2}\left(\frac{A_D}{A_L + A_D}\right)\frac{(\delta_z + \epsilon\Delta_x)}{\Delta_x}\cos\psi \tag{B-1}$$

and

$$\ddot{\beta} + d\,\dot{\beta} + \Omega^2 \beta - R_x\,(\dot{\alpha} + c\alpha) = -\,R_x\,\dot{\phi}_{ss} + \omega_n^{\,2}\left(\frac{A_D}{A_L + A_D}\right)\frac{(\delta_z + \epsilon\Delta_x)}{\Delta_x}\sin\psi \tag{B-2}$$

In Appendix A, the homogeneous solutions of Eq. (B-1) and (B-2) are examined. Once the homogeneous solution of a linear differential equation has been determined, it is always possible to find the complete solution, at least in an integral form. In the present case, the integrals involved in determining the forced, or particular, solutions are very complex, and as a result. approximate methods must be utilized.

One of the first approximations that might be made is to neglect the derivative terms that occur in Eq. (B-1) and (B-2). An examination of the various forcing functions of Eq. (B-1) and (B-2) indicates that the forced or 'steady state' solutions should contain primarily very low frequency components. Thus

$$\alpha_{ss}\Omega^2 + R_x\,c\,\beta_{ss} = \ddot{\phi}_{ss} + K_M\,\dot{\phi}_{ss} + \omega_n^{\,2}\left(\frac{A_D}{A_L + A_D}\right)\frac{(\delta_z + \epsilon\Delta_x)}{\Delta_x}\cos\psi \tag{B-3}$$

and

$$\beta_{ss}\Omega^2 - R_x\,c\,\alpha_{ss} = -\,R_x\,\dot{\phi}_{ss} + \omega_n^{\,2}\left(\frac{A_D}{A_L + A_D}\right)\frac{(\delta_z + \epsilon\Delta_x)}{\Delta_x}\sin\psi \tag{B-4}$$

Solving Eq. (B-3) and (B-4) for α_{ss} and β_{ss}, and simplifying yields, respectively

$$\alpha_{ss} = \frac{\Omega^2 \left[\ddot{\phi}_{ss} + K_M \dot{\phi}_{ss} + \omega_n^2 \left(\frac{A_D}{A_L + A_D}\right) \frac{(\delta_z + \epsilon \Delta_x)}{\Delta_x} \cos \psi \right] + R_x^2 c \dot{\phi}_{ss}}{\Omega^4 + R_x^2 c^2}$$

$$- \frac{R_x c \omega_n^2 \left(\frac{A_D}{A_L + A_D}\right) \frac{(\delta_z + \epsilon \Delta_x)}{\Delta_x} \sin \psi}{\Omega^4 + R_x^2 c^2} \qquad \text{(B-5)}$$

and

$$\beta_{ss} = \frac{\Omega^2 \left[- R_x \dot{\phi}_{ss} + \omega_n^2 \left(\frac{A_D}{A_L + A_D}\right) \frac{(\delta_z + \epsilon \Delta_x)}{\Delta_x} \sin \psi \right] + R_x c \ddot{\phi}_{ss} + R_x c K_M \dot{\phi}_{ss}}{\Omega^4 + R_x^2 c^2}$$

$$+ \frac{R_x c \omega_n^2 \left(\frac{A_D}{A_L + A_D}\right) \frac{(\delta_z + \epsilon \Delta_x)}{\Delta_x} \cos \psi}{\Omega^4 + R_x^2 c^2} \qquad \text{(B-6)}$$

For the usual case in which the term representing the static moment is predominant, Eq. (B-5) and (B-6) reduce to the following form:

$$\alpha_{ss} = \frac{\ddot{\phi}_{ss}}{\omega_n^2} + \frac{K_M \dot{\phi}_{ss}}{\omega_n^2} + \left(\frac{A_D}{A_L + A_D}\right) \frac{(\delta_z + \epsilon \Delta_x)}{\Delta_x} \cos \psi \quad . \qquad \text{(B-7)}$$

and

$$\beta_{ss} = -\frac{R_x \dot{\phi}_{ss}}{\omega_n^2} + \left(\frac{A_D}{A_L+A_D}\right)\frac{(\delta_z + \epsilon\Delta_x)}{\Delta_x} \sin \psi \tag{B-8}$$

By differentiating Eq. (B-7) and (B-8) and introducing the results into Eq. (B-1) and (B-2), a second approximation for α_{ss} and β_{ss} may be obtained. However, unless the static margin of the vehicle is very close to zero, Eq. (B-7) and (B-8) are excellent representations of the forced solutions in the altitude range from 250,000 ft to sea level.

From the definition of ω_n^2, $\dot{\phi}_{ss}$ given on page 8 , and of R_x given on page 31 we find for α_{ss} and β_{ss}

$$\alpha_{ss} = \frac{-\frac{d}{dt}\left(\frac{g}{V}\cos\phi_{ss}\right)}{\frac{\Delta_x}{I}(A_L+A_D)V^2} e^{-kh} - \frac{K_M g \cos\phi_{ss} e^{-kh}}{\frac{\Delta_x}{I}(A_L+A_D)V^3} + \left(\frac{A_D}{A_L+A_D}\right)\frac{(\delta_z + \epsilon\Delta_x)}{\Delta_x}\cos\psi \tag{B-9}$$

and

$$\beta_{ss} = \frac{I_x \omega_x g \cos\dot{\phi}_{ss} e^{-kh}}{\Delta_x (A_L+A_D) V^3} + \left(\frac{A_D}{A_L+A_D}\right)\frac{(\delta_z + \epsilon\Delta_x)}{\Delta_x}\sin\psi \tag{B-10}$$

From Eq. (B-9) and (B-10), it can be seen that as Δ_x approaches zero, the steady-state values of α and β approach infinity. But as Δ_x goes to zero, ω_n^2 becomes very small and the original approximation is no longer valid since the derivative terms become significant.

For the special case in which Δ_x is zero, the exact forced solutions for α and β may be deduced from Eq. (B-1) and (B-2). Thus

$$\alpha_{ss} = e^{-\int c\,dt}\int e^{\int c\,dt}\, d\,\phi_{ss} \tag{B-11}$$

and

$$\beta_{ss} = 0 \tag{B-12}$$

Upon considering Eq. (B-10) and (B-12), it appears that as Δ_x becomes small, initially the forced solutions increase in magnitude until the point is reached where the derivative terms become important. From that point until Δ_x is zero, the magnitude of the forced terms decrease.

APPENDIX C

The Aerodynamic Parameters

On page 7 the lift and drag forces are defined in terms of two parameters, A_L and A_D. These parameters, as indicated by Eq. (21) and (22), are functions of the sea level air density, a body reference area, and the lift curve slope and drag coefficient, respectively.

Thus

$$\begin{aligned} L_\alpha &= A_L\, e^{kh}\, V^2 \alpha && \text{(a)} \\ L_\beta &= A_L\, e^{kh}\, V^2 \beta && \text{(b)} \\ D &= A_D\, e^{kh}\, V^2 && \text{(c)} \end{aligned} \qquad \text{(C-1)}$$

where

$$A_L = \frac{1}{2} \rho_o A_{ref.} C_{L_\alpha}$$

$$A_D = \frac{1}{2} \rho_o A_{ref.} C_D$$

The damping coefficient, K_M, may be defined in a similar manner. Thus

$$K_M = A_M V e^{kh} \qquad \text{(C-2)}$$

where

$$A_M = -\frac{(C_{m_q} + C_{m_{\dot{\alpha}}})}{2\,I} \rho_o A_{ref.} \ell^2_{ref.}$$

and $\ell_{ref.}$ is the length of the body.

If a conical re-entry body is considered, then at hypersonic speeds impact theory yields the following expressions for C_L , C_D, C_{m_q} and $C_{m_{\dot{\alpha}}}$:*

$$\left. C_{L_\alpha} \right|_{\alpha \to 0} = 2 \left(\cos^2 B - \sin^2 B\right) \qquad \text{(a)}$$

$$C_D = 2 \sin^2 B \qquad \text{(b)} \qquad \text{(C-3)}$$

$$\left. C_{m_q} \right|_{q \to 0} = - \frac{1}{\cos^2 B} + \frac{8}{3} \left(\frac{\ell_{cg}}{\ell_{ref}}\right) - 2 \cos^2 B \left(\frac{\ell_{cg}}{\ell_{ref}}\right)^2 \qquad \text{(c)}$$

$$\left. C_{m_{\dot{\alpha}}} \right|_{\dot{\alpha} \to 0} = 0 \qquad \text{(d)}$$

where ℓ_{cg} is the distance from the nose of the cone to the center of gravity and B is the cone half-angle. In terms of the static margin, Δ_x, we find for $\frac{\ell_{cg}}{\ell_{ref}}$:

$$\frac{\ell_{cg}}{\ell_{ref}} = \frac{\Delta_x}{\ell_{ref}} + \frac{2}{3} \frac{1}{\cos^2 B} \qquad \text{(C-4)}$$

Utilizing Eq. (C-3) and (C-4) we find for A_L, A_D and A_M

$$A_L = \rho_o A_{ref} \left(\cos^2 B - \sin^2 B\right) \qquad \text{(C-5)}$$

$$A_D = \rho_o A_{ref} \sin^2 B \qquad \text{(C-6)}$$

$$A_M = \frac{\rho_o A^2_{ref}}{18 \pi I \sin^2 B} + \frac{\rho_o \Delta_x^2 A_{ref}}{I} \cos^2 B \qquad \text{(C-7)}$$

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*See Ref. 2.

The loading parameter, N, is defined as follows

$$N = \frac{mg}{A_{ref} C_D} = \frac{mg}{2\, A_{ref}\, \sin^2 B} \tag{C-8}$$

By substituting Eq. (C-8) into Eq. (C-5)-(C-7), the reference area may be eliminated. We are now in a position to evaluate the relative magnitudes of the coefficients of Eq. (A-1) and (A-2). Thus:

1) The damping term

$$d = \frac{A_L}{m} V e^{kh} + K_M = V e^{kh} \left[\frac{\rho_o g}{2N} (\cot^2 B - 1) + \frac{\rho_o (mg)^2}{72 \pi I N^2 \sin^6 B} + \frac{\rho_o \Delta_x^2 mg}{2 N I} \cot^2 B \right] \tag{C-9}$$

2) The undamped natural frequency

$$\Omega^2 = \omega_h^2 + \frac{d}{dt} \left(\frac{A_L}{m} V e^{kh} \right) + K_M \frac{A_L}{m} V e^{kh}$$

$$= - \frac{\Delta_x}{I} \frac{\rho_o mg}{2N} \cot^2 B\, V^2 e^{kh} + \frac{\rho_o g}{2N} (\cot^2 B - 1) \left[\dot{V} e^{kh} + k\dot{h} V e^{kh} \right] \tag{C-10}$$

$$+ \left[\frac{\rho_o g}{2N} (\cot^2 B - 1) \right] \left[\frac{\rho_o (mg)^2}{72 \pi I N^2 \sin^6 B} + \frac{\rho_o \Delta_x^2 mg}{2 N I} \cot^2 B \right] V^2 e^{2kh}$$

The expression for Ω^2 may be further simplified by recognizing the following relationships:

$$\dot{V} = - \frac{A_D}{m} V^2 e^{kh} + g \sin \phi_{ss}$$

and

$$\dot{h} = - V \sin \phi_{ss}$$

3) The aerodynamic coupling term

$$c = \frac{A_L}{m} V e^{kh} = \frac{\rho_o g}{2N} (\cot^2 B - 1) V e^{kh} \qquad \text{(C-11)}$$

An inspection of Eq. (C-9)-(C-11) indicates that for a cone half-angle of 45 deg, the expressions for d, Ω^2 and C are greatly simplified. This is due to the fact that impact theory yields a lift curve slope equal to zero for such a case.

The comparative magnitudes of the three terms which constitute Ω^2 may be seen in the following table for a typical re-entry cone:

Table C-1

$\Delta_x = -0.025$ ft	$\frac{2N}{\rho_o g V^2 e^{kh}} \omega_n^2$	$\frac{2N}{\rho_o g V^2 e^{kh}} \frac{d}{dt}\left(\frac{A_L}{m} V e^{kh}\right)$	$\frac{2N}{\rho_o g V^2 e^{kh}} K_M \frac{A_L}{m} V e^{kh}$
B = 30 deg			
N = 140			
m = 100 slugs			
h = 500,000 ft	4.69×10^{-2}	2.46×10^{-5}	5.57×10^{-12}
250,000	4.69×10^{-2}	2.46×10^{-5}	1.57×10^{-7}
50,000	4.69×10^{-2}	1.52×10^{-5}	5.73×10^{-4}

From Table C-1 it can be seen that even with a static margin of 0.3 of an inch, the ω_n^2 term is, under the least favorable circumstances, larger than the remaining terms by a factor on the order of 100. Under the same conditions, the damping term, d, and the coupling term, c, are roughly one-tenth the magnitude of ω_n^2. Thus terms containing the square of c and d, or their product, which occur in Eq. (A-28) may be neglected as compared to ω_n^2. A negative increase in Δ_x serves to increase the disparity in magnitudes between ω_n^2 and the other terms.

APPENDIX D

Planar Re-entry Analysis

The similarity between the planar solution obtained in Appendix A and the results obtained by other investigators* is not immediately obvious. By solving the problem with altitude rather than time as the independent variable, the degree of equivalence of Eq. (A-20) to the results of Ref. 2 becomes more apparent.

Noting that

$$\frac{dh}{dt} = - V \sin \phi_{ss} \tag{D-1}$$

Eq. (A-3) may be transformed to the following form

$$\begin{aligned} \frac{d^2\alpha}{dh^2} &+ \frac{d\alpha}{dh}\left[\frac{A_D\, e^{kh}}{m \sin \phi_{ss}} - \frac{g}{V^2 \sin^2 \phi_{ss}} - \frac{\left(A_L/_m + A_M\right)}{\sin \phi_{ss}} e^{kh}\right] \\ &+ \left\{- \frac{\Delta_x}{I} \frac{(A_L + A_D)}{\sin^2 \phi_{ss}} e^{kh} + \frac{1}{V^2 \sin^2 \phi_{ss}} \frac{d}{dt} \left(\frac{A_L}{m} V\, e^{kh}\right) \right. \\ &\left. + \frac{A_M\, A_L}{m \sin^2 \phi_{ss}} e^{2kh}\right\}\alpha = 0 \end{aligned} \tag{D-2}$$

By the method utilized in Appendix A, we find

$$\omega^2 (h) \cong - \frac{\Delta_x}{I} \frac{(A_L+A_D)}{\sin^2 \phi_{ss}} e^{kh} + \frac{3}{4}\left[\frac{\omega'(h)}{\omega(h)}\right]^2 - \frac{1}{2}\frac{\omega''(h)}{\omega(h)} \tag{D-3}$$

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*See page 8, Ref. 2.

where the second-order terms have been neglected. As a first approximation,

$$\omega_1^{\,2}(h) = -\frac{\Delta_x}{I}\frac{(A_L+A_D)}{\sin^2\phi_{ss}}e^{kh} \tag{D-4}$$

Substituting $\omega_1(h)$ and $\omega_1'(h)$ into Eq. (D-3) indicates that Eq. (D-4) is a good approximation as long as

$$-\frac{\Delta_x}{I}\frac{(A_L+A_D)}{\sin^2\phi_{ss}}e^{kh} \gg \frac{k^2}{16} \tag{D-5}$$

For a Δ_x of -0.5 ft or larger (in a negative sense), the inequality of Eq. (D-5) is valid for altitudes from 250,000 ft to sea level.

If the $\frac{g}{V^2\sin^2\phi_{ss}}$ term of Eq. (D-2) is neglected we find for $\sigma(h)$:

$$\sigma(h) = -\frac{1}{2}\left[\frac{A_D\, e^{kh}}{m\sin\phi_{ss}} - \frac{\left(\frac{A_L}{m}+A_M\right)e^{kh}}{\sin\phi_{ss}}\right] - \frac{1}{2}\frac{\omega'(h)}{\omega(h)} \tag{D-6}$$

Thus for α

$$\alpha = \frac{e^{-\frac{1}{2}\left[\frac{\frac{A_D}{m}-\frac{A_L}{m}-A_M}{k\sin\phi_{ss}}\right]e^{kh}}}{\left[\frac{\Delta_x}{I}\frac{(A_L+A_D)}{\sin^2\phi_{ss}}e^{kh}\right]^{1/4}}\left\{c_1\, e^{j\int\omega(h)\,dh} + c_2\, e^{-j\int\omega(h)\,dh}\right\} \tag{D-7}$$

where ϕ_{ss} is assumed to be a constant. A comparison of Eq. (D-7) with Eq. (15e) of Ref. 2 reveals that the terms which affect the amplitude of the oscillation are identical except for a constant which may be absorbed in c_1 and c_2.

In Ref. 2, the point is made that for a high-drag body, it would be possible for the exponent of Eq. (D-7) to be positive, and thus for divergence to occur. However, an examination of the origin of the drag term which occurs in the exponent of the exponential in conjunction with the implicit assumptions that have been made by neglecting $\frac{g}{V^2 \sin^2 \phi_{ss}}$ term of Eq. (D-2), indicates that divergence due to a high-drag configuration is not a serious problem.

As an intermediate step in the transformation of Eq. (A-3) to Eq. (D-2) by the use of Eq. (D-1), we find, for the coefficient of $\frac{d\alpha}{dh}$, the following expression:

$$\left(\frac{1}{V}\frac{dV}{dh} + \cot \phi_{ss} \frac{d\phi_{ss}}{dh} - \frac{d}{V \sin \phi_{ss}}\right) .$$

Under the assumptions of Ref. 2, ϕ_{ss} is a constant, thus $d\phi_{ss}/dh$ is zero, and the speed as a function of altitude is approximated by

$$V = V_o \, e^{\frac{A_D}{m\,k \sin \phi_{ss}} e^{kh}} \tag{D-8}$$

where the component of g along the flight path has been neglected. With these approximations, the coefficient of $\frac{d\alpha}{dh}$ reduces to the form used in Eq. (D-6). However, the method of solution adopted in this report does not require the above approximations.

Thus

$$\sigma(h) = -\frac{1}{2}\left[\frac{1}{V}\frac{dV}{dh} + \cot \phi_{ss} \frac{d\phi_{ss}}{dh} - \frac{\left(\frac{A_L}{m} + A_M\right)}{\sin \phi_{ss}} e^{kh}\right] - \frac{1}{2}\frac{\omega'(h)}{\omega(h)} \tag{D-9}$$

and

$$\alpha = \frac{e^{\frac{\left(\frac{A_L}{m} + A_M\right)e^{kh}}{2\,k\,\overline{\sin \phi}_{ss}}}}{\left[-\frac{\Delta_x}{I}\,(A_L + A_D)\,e^{kh}\right]^{1/4}} \left[\frac{V_o \sin \phi_c}{V}\right]^{-1/2} \left\{c_1\, e^{j\int \omega(h)\,dh} + c_2\, e^{-j\int \omega(h)\,dh}\right\} \qquad \text{(D-10)}$$

Unless the lift curve slope is negative, the exponent of the exponential term of Eq. (D-10) is never positive. The ratio $\left(\frac{V_o}{V}\right)^{1/2}$ of Eq. (D-10) corresponds to the exponential drag term of Eq. (D-7). Although V can become quite small compared to V_o, it never approaches zero. The terminal speed of almost any conceivable re-entry body would be on the order of 500 ft/sec. In Eq. (D-7), the component of g along the flight path has been neglected, and as a consequence, V may approach zero, and the amplitude of α may thus approach infinity. Considering just the envelope of the oscillation, as given by Eq. (D-10), we have

$$\left(\frac{\alpha}{\alpha_o}\right)_{env.} = e^{\frac{\left(\frac{A_L}{m} + A_M\right)}{2\,k\,\overline{\sin \phi}_{ss}}\left(e^{kh} - e^{kh_o}\right)} \times \left(\frac{V_o}{V}\right)^{1/2} \frac{e^{\frac{kh_o}{4}}}{e^{\frac{kh}{4}}} \qquad \text{(D-11)}$$

Equation (A-20) of Appendix A is equivalent to Eq. (D-11) if the initial conditions are established at an altitude of 200,000 ft where the k^2 term is small compared to ω_n^2. At higher altitudes, where ω_n^2 becomes very small we find for the envelope:

$$\left(\frac{\alpha}{\alpha_o}\right)_{env.} = \frac{\sqrt{-k\, V_o \sin \phi_o}\; e^{\frac{\left(\frac{A_L}{m} + A_M\right)}{2\,k\,\overline{\sin \phi}_{ss}}\left(e^{kh} - e^{kh_o}\right)}}{2\left[-\frac{\Delta_x}{I}\left(A_L + A_D\right) V^2 e^{kh} + \frac{1}{16} k^2 V^2 \sin^2 \phi\right]^{1/4}} \tag{D-12}$$

Figure 10 is a plot of $\left(\frac{\alpha}{\alpha_o}\right)_{env.}$ as a function of altitude.

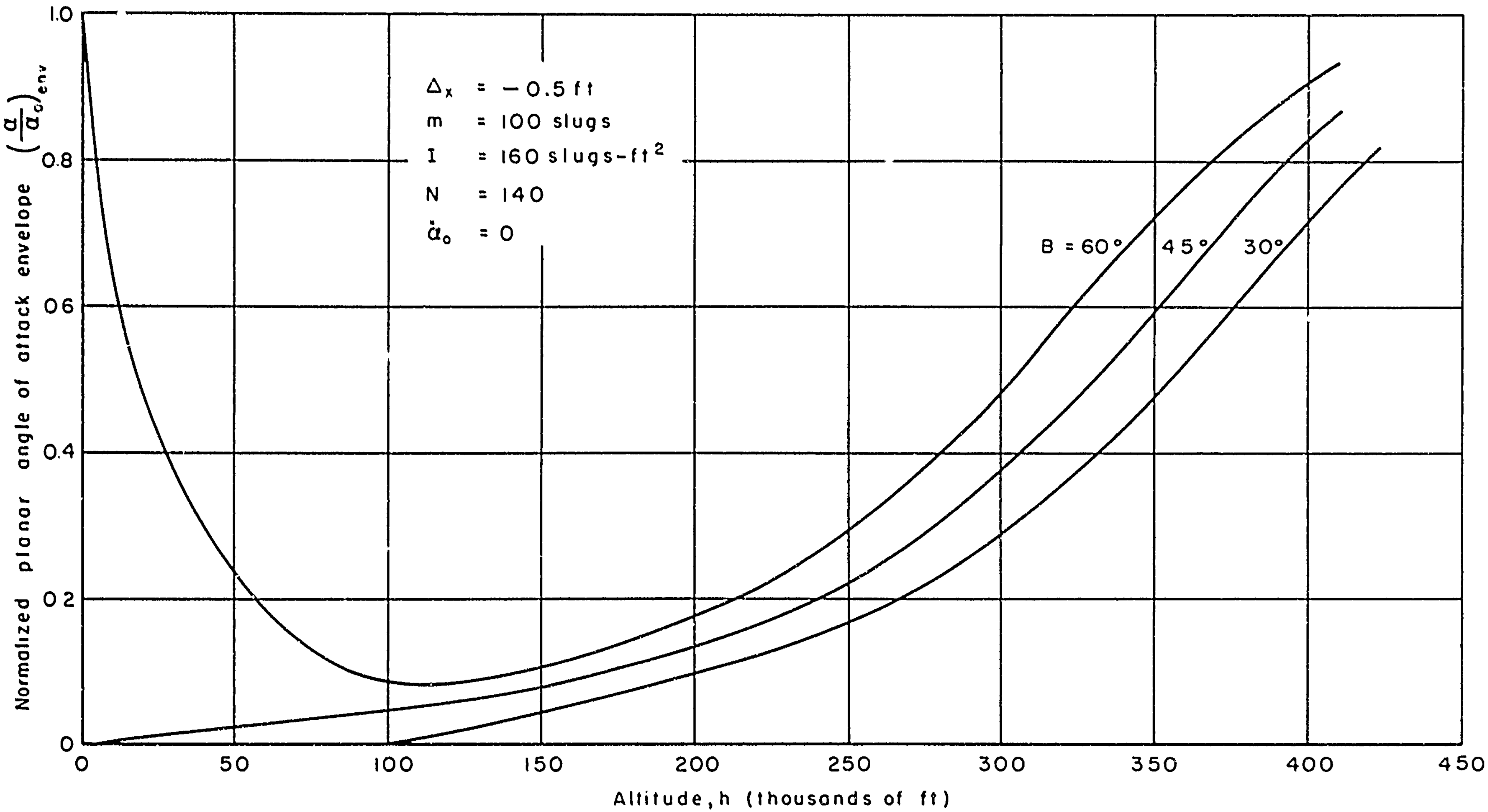

Fig. 10 — Planar angle of attack envelope as a function of altitude

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